# ORTHOTROPIC SHELL ANALYSIS OF A SHIPS BOTTOM SECTION

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Lieutenant, United States Navy
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SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF NAVAL ENGINEER
AND THE DEGREE OF
MASTER OF SCIENCE IN NAVAL ARCHITECTURE
AND MARINE ENGINEERING
at the
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19 May 1962

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## ABSTRACT

In the interest of developing more thorough methods of analyzing ship structures, an entire bottom section extending from side to side and from one bulkhead to another is analyzed as an orthotropic shell of variable curvature. The longitudinally framed destroyer type ship is studied under hydrostatic loading and longitudinal bending stress. Specific consideration is given to the orthotropy of the structure and its curvature.

As a result of the combined analysis of the shell and its associated stiffener system, partial differential equations are developed which define the behavior of the stiffened shell. The inherent complexity of the developed equations precludes an analytic solution. An approximate solution could be obtained, however, by the use of difference equations and digital machine computation.

An application of membrane theory to an equivalent orthotropic shell provides a means of determining direct stresses and extensional rigidities. A comparative study of two existing destroyer types using developed theory reveals that membrane theory with corrective iterations is of value in preliminary analysis. The sensitivity of the variation of hull curvature from the analytic curvature is also demonstrated. The use of the midship section coefficient as a design parameter for the structural designer should be encouraged and exploited. If the midship section area is kept constant the curvature can be varied to the advantage of the structure with no effect on speed. The relatively high bending stiffness in the girthwise direction of current designs merits a reevaluation of transverse strength requirements.

Structural design procedures employing orthotropic rigidity coefficients in combination with either the solution to the above differential equations or a modified membrane theory, have been outlined by the author.

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# NOTATION

$b_1$ = transverse web frame spacing (in.)

$b_2$ = longitudinal stringer spacing (in.)

$d_1$ = $\dfrac{\partial}{\partial x} = (\quad)'$

$d_2$ = $\dfrac{\partial}{\partial \phi} = \dot{(\quad)}$

$d_1^{\,n}$ = $\dfrac{\partial^n}{\partial x^n}$

$d_2^{\,n}$ = $\dfrac{\partial^n}{\partial \phi^n}$

p = external load ($lb/ft^2$)

q = hydrostatic load ($lb/ft^2$)

r = radius of curvature (ft)

t = shell thickness (in.)

$t_e$ = equivalent shell thickness (in.)

u = axial displacement

v = girthwise displacement

w = radial displacement

x = longitudinal coordinate (ft.)

z = distance from shell middle surface

A = cross sectional area of a stiffening member ($in.^2$)

B = beam of ship at design waterline (ft.)

C = dist. of centroid of stiffening member from shell middle surface (in.)

D = extensional rigidity (lb/in)

$E$ = modulus of elasticity (psi)

$G$ = shear modulus (psi)

$H$ = draft of ship at design waterline (ft.)

$I$ = moment of inertia of stiffening member about its N. A. ($in^4$)

$J$ = polar moment of inertia ($in^4$)

$K$ = bending rigidity coefficient (lb - in)

$L$ = length of shell between bulkheads (ft.)

$L_s$ = length of ship (ft.)

$M$ = moment resultant (lb-ft/ft)

$N$ = force resultant (lb/ft)

$Q$ = transverse shear force resultant (lb/ft)

$S$ = rigidity moment coefficient (lb-in)

$V_B$ = transverse vertical edge loading on shell (lb/ft)

$V_L$ = longitudinal vertical edge loading on shell (lb/ft)

$\gamma$ = shear strain

$\epsilon$ = linear strain

$\nu$ = Poissons ratio

$\sigma$ = stress intensity (psi)

$\sigma_y$ = elastic yield stress (psi)

$\sigma_1$ = longitudinal hull girder bending stress (psi)

$\tau$ = shear stress (psi)

$\phi$ = girthwise or circumferential coordinate (deg)

# I. INTRODUCTION

This analysis can be considered as a contribution to the continuing effort being made to produce more rational and less empirical ship structural design procedures. The analysis of a structure whose loading and response characteristics are not readily defineable does indeed present a challange. There exists, however, opportunities for improvement in the static, elastic analysis of ship structures. Such improvements could contribute towards a better determination of adequate but not excessive scantlings arranged in the most efficient manner. Scantlings are normally selected on the basis of a series of approximate analyses. While the effect of continuity is often considered, the effects of adjoining members and shell curvature are neglected. Checks are made on the adequacy of stiffeners and stiffener-plating combinations on a panel basis. This artificial method of analysis, while adequate and sufficiently approximate in the past, cannot be considered as satisfactory for the future. The knowledge of ship seaway loadings is increasing. Advances are being made in dynamic loading theory. If these accomplishments are eventually to be used to greatest advantage in design, they must be used in conjunction with a realistic static analysis.

The structural analysis of a plate or shell stiffener system can be approached in general by either of two basically different view points. The first is that of a grillage beam analysis

wherein the plating is taken to act as additional flanges to the stiffeners in an orthogonal network. Essentially the grillage beam concept as originally developed by Vedeler (1)* relys on equating deflections at stiffener intersections and moment distribution. The solutions may be obtained by orthodox solutions to differential equations, trigonometric series, energy methods or relaxation and numerical methods suitable for digital machine computation. Variations in the grillage beam technique and its solutions have also been developed (2). One need only to consult contemporary technical journals relating to aeronautical and civil engineering to find a prolific assortment of techniques and methods for grillage beam solutions most of which lead ultimately to machine computation (e.g. stiffness matrices).

The second basic approach is to utilize plate or shell theory for the given structure in such a manner that the stiffener system effect is incorporated into the medium. This is accomplished by substituting for the actual shell with stiffeners (frames and longitudinals) an equivalent shell without stiffeners which exhibits different elastic properties in the longitudinal and circumferential directions (orthotropic behavior). The manner in which the equivalent orthotropic shell is determined will be discussed shortly. Let it suffice at this point to state that the result of such a substitution will be to obtain a shell structure whose "general" behavior

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* Numbers in brackets after an author's name indicate references listed in Appendix E of this paper.

under loading is equivalent to that of the stiffened shell. The orthotropic behavior ofthe equivalent shell may be called constructional since it does not result from a property of the material but from the manner in which it is distributed. It is interesting to note at this point that this orthotropic approach is diametrically opposite to the grillage beam method (i.e.,the grillage beam assumes plating to act as additional flange material for stiffeners whereas in the orthotropic shell, the stiffener influence is distributed to the shell or plating). It is felt that the orthotropic shell concept has sufficient merit to be the general approach used in this analysis. Physically it appears to be more meaningful in the response and behavior of the stiffened structure. While the task of analysis of such a structure is not simple, it is at least amenable to a mathematical investigation. Orthotropic shell theory consistent with the structure being studied will be utilized.

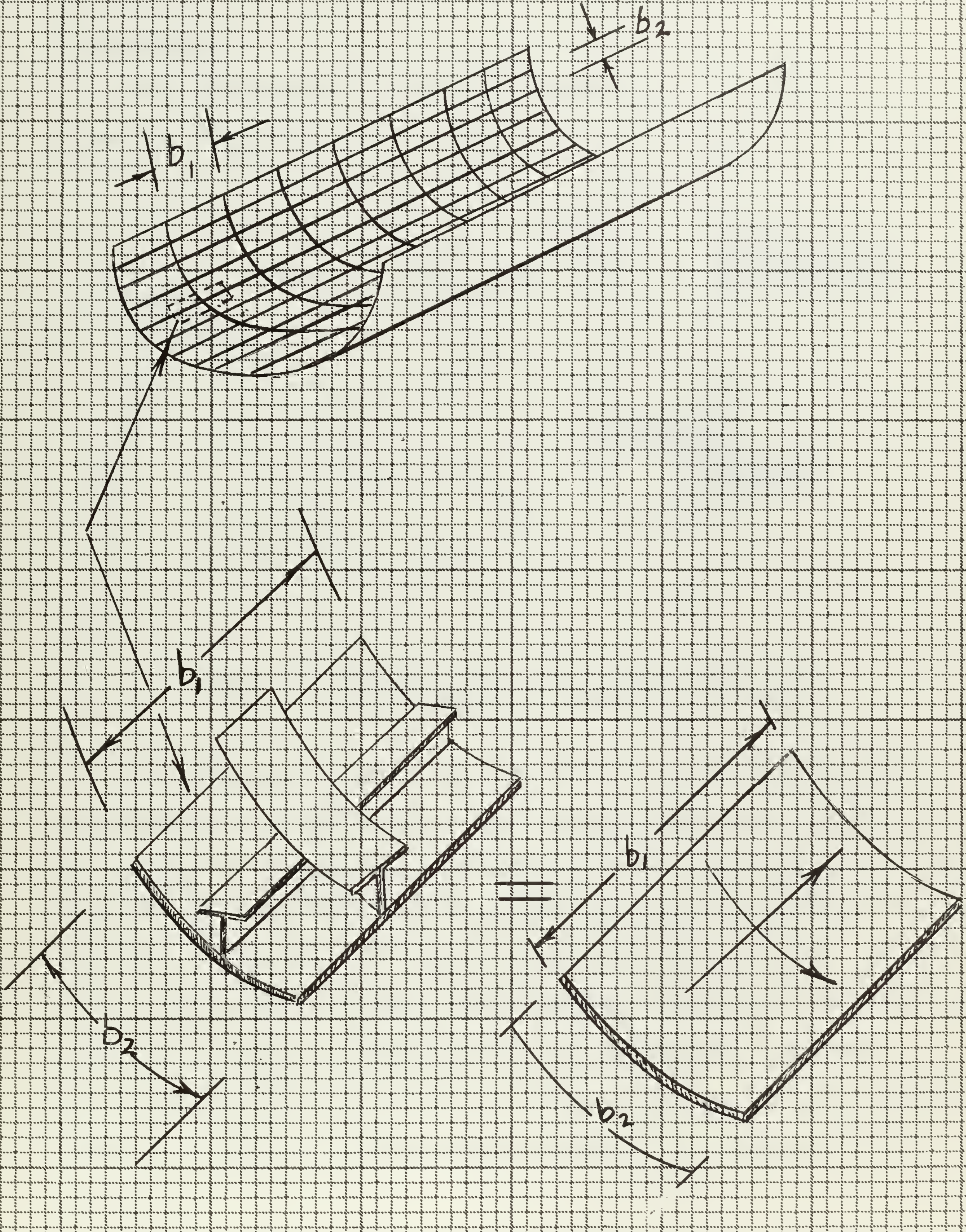
Additional methods of shell or plate stiffener systems include the concept that the stiffener system acts as an elastic support for the loading and as such may be treated by an application of the beam on an elastic foundation method (3). Another interesting technique is to disseminate the effect of one set of stiffeners (frames) into the plating while other (longitudinals) set is subjected to a harmonic analysis producing influence coefficients for deflection or bending moment for each longitudinal (4).

This analysis will investigate the structural behavior of the bottom section of a ship from side to side and from bulkhead

to bulkhead. The ship type considered will be a longitudinally framed naval warship in the destroyer catagory. Emphasis will be given to shell curvature, continuity of the shell between bulkheads, and the stiffener characteristics. Normal hydrostatic loading combined with the overall longitudinal bending stress will be considered. Shade (5) developed a bending theory for ships bottom structure utilizing orthotropic plate theory. However, a flat bottom was assumed and no consideration was given to hull curvature.

Figure 1. Orthotropic element of

SHIP STRUCTURE



## II. ANALYSIS

### Preliminary

The portion of the ships structure to be analyzed is shown in Figure 2.

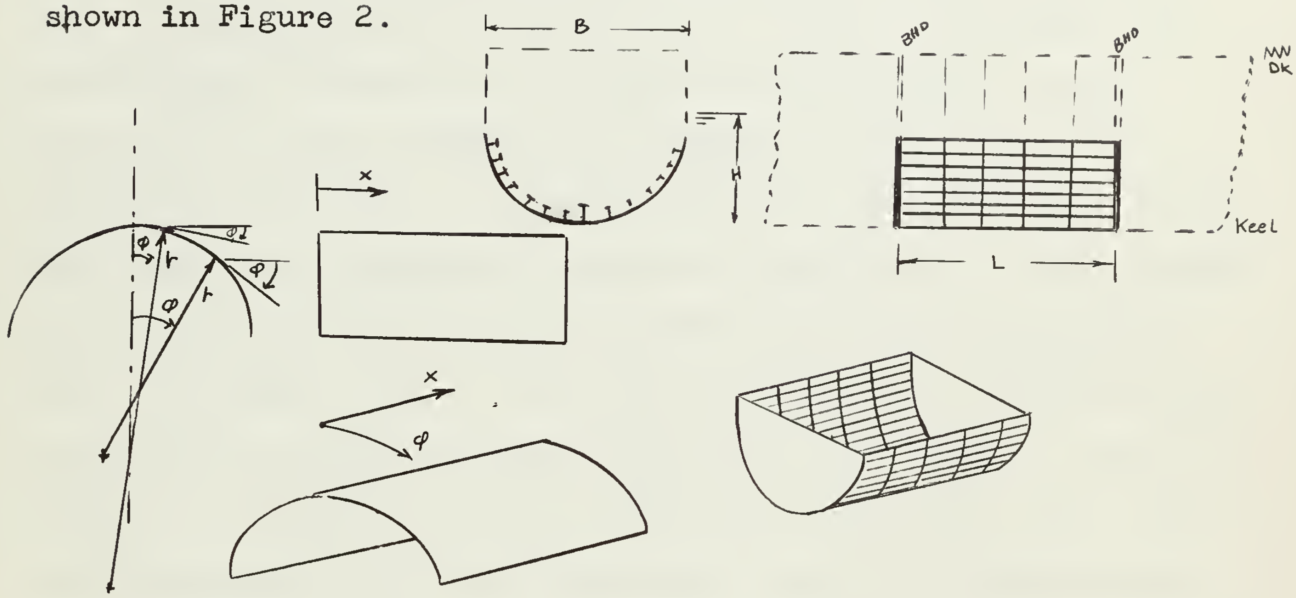


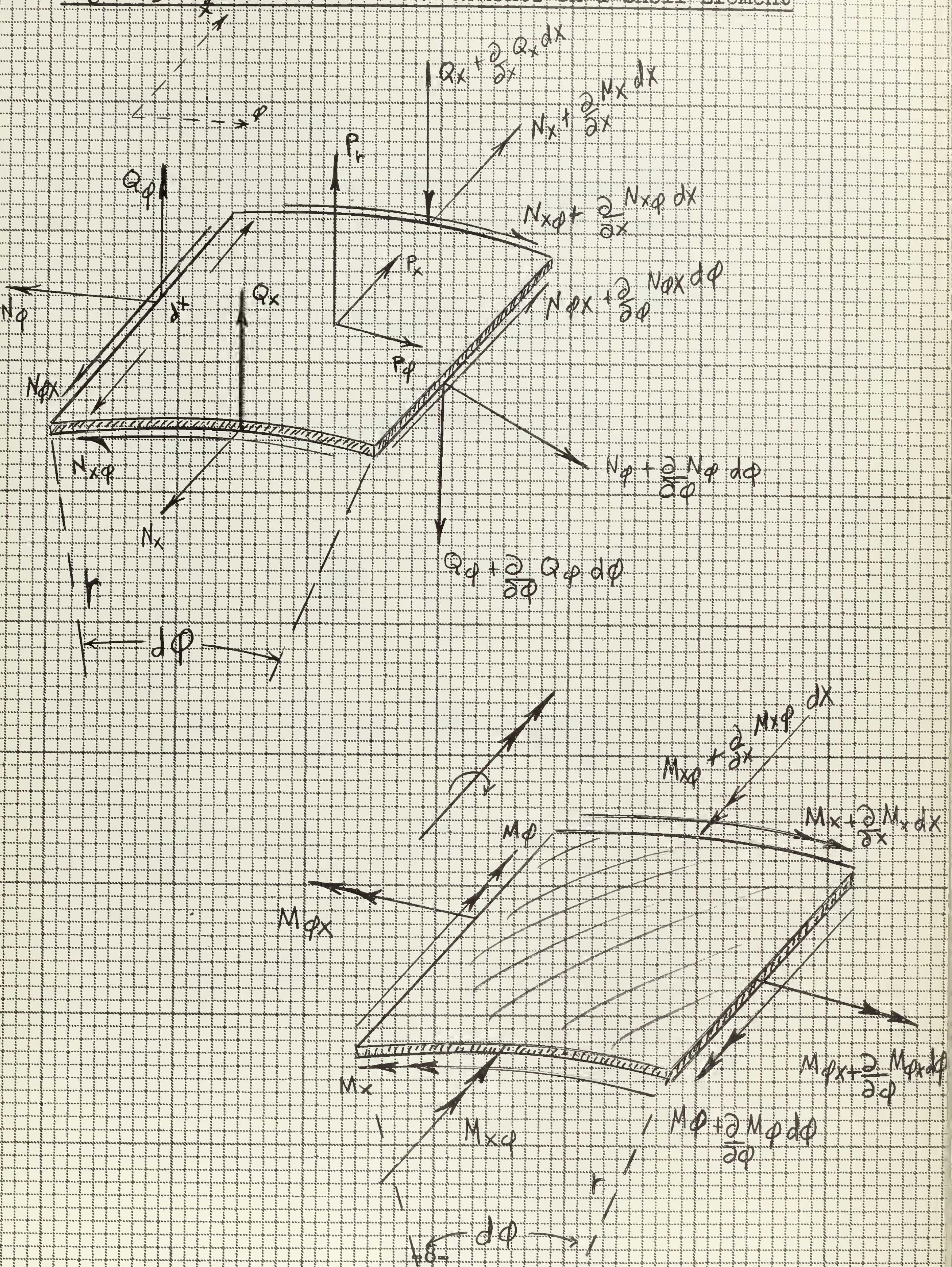

Fig. 2 Identification of Section Analyzed

The boundary conditions for this section are discussed in Appendix C . In brief they are approximated as follows: The degree of fixity can be assumed to be 100% clamped at the junction of the shell and bulkhead and simply supported at the junction of the curved shell to the vertical or flat portion of the ships side. While these assumed conditions are not in reality present, a determination of the elastic degree of fixity is outside the realm of this analysis. The postulated conditions will be sufficiently accurate for approximation purposes.

The inclusion of the hull curvature in the analysis can be

effected by the selection of an analytical expression for the radius of curvation (r), as a function of girth angle ($\phi$). The selection of the expression for the radius of curvature must be uncomplicated enough so that it will not impede the mathematical operations involved in the analysis. At the same time it must sufficiently describe the hull form for any given B/H ratio. A consideration of pertinent hull forms and analytical expressions is made in Appendix A. The resulting analytical expression satisfying the above requirements has been found to be $r = 2H\cos\phi$; the radius of curvature which describes a cycloid of depth H and span $\pi$ H. One of the significant contentions of this thesis is that a ship type such as a destroyer has a hull form which cannot be treated with flat bottom or flat plate assumptions. Of course if a restriction is made to only a small panel of the structure, a flat surface is more closely approached. This type of panel analysis, however, is a rather limited view of the structures behavior and disallows for the continuity of deformation. Moreover, not only is significant curvature present but it is variable around the circumference of the hull. An assumption of constant curvature or a circular cylindrical shell would greatly facilitate the mathematical analysis but would not yield results which are representative of the structure. It is felt that the variable curvature selected represents sufficiently the type of hull forms being considered (see Figs. 19-23).

Figure 3 System of Force and Moments on a Shell Element



## Isotropic Shell Analysis

To begin the analysis the elastic laws for the variable curvature isotropic shell (ships hull without stiffener system) will be developed. The shell element and associated stress system are indicated in Fig. 3.

Equilibrium of forces provide the following relations:

x dir:

$$\frac{\partial (N_x)}{\partial x} dxrd\phi + \frac{\partial (N_{\phi x})}{\partial \phi} d\phi dx + p_x rd\phi dx = 0$$

or

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{\phi x}}{\partial \phi} + p_x = 0 \qquad (1)$$

$\phi$ dir:

$$\frac{\partial (N_\phi)}{\partial \phi} d\phi dx \quad \frac{\partial (N_{x\phi})}{\partial x} dxrd\phi - Q_\phi dx\frac{d\phi}{2} - Q_\phi dx\frac{d\phi}{2}$$

$$- \frac{\partial (Q_\phi)}{\partial \phi} d\phi dx\frac{d\phi}{2} \quad p_\phi rd\phi dx = 0$$

or

$$\frac{1}{r}\frac{\partial N_\phi}{\partial \phi} \quad \frac{\partial N_{x\phi}}{\partial x} - \frac{1}{r} Q_\phi + p_\phi = 0 \qquad (2)$$

r dir:

$$- 2N_\phi dx\frac{d\phi}{2} - \frac{\partial(Q_x)}{\partial x}dxrd\phi - \frac{\partial}{\partial\phi}(Q_\phi)d\phi dx + p_r dxrd\phi = 0$$

or

$$\frac{1}{r} N_\phi + \frac{\partial Q_x}{\partial x} + \frac{1}{r}\frac{\partial Q_\phi}{\partial\phi} - p_r = 0 \qquad (3)$$

Equilibrium of moments give —

about $p_x$ axis:

$$-\frac{\partial(M_{x\phi})}{\partial x} dxrd\phi - \frac{\partial(M_\phi)}{\partial\phi}d\phi dx + Q_\phi dxrd\phi = 0$$

or

$$\frac{\partial M_{x\phi}}{\partial x} + \frac{1}{r}\frac{\partial M_\phi}{\partial\phi} - Q_\phi = 0 \qquad (4)$$

about $p_\phi$ axis:

$$\frac{\partial}{\partial x}(M_x)dxrd\phi + \frac{\partial(M_{\phi x})}{\partial\phi}d\phi dx - Q_x dxrd\phi = 0$$

or

$$\frac{\partial M_x}{\partial x} \quad \frac{1}{r}\frac{\partial M_{\phi x}}{\partial\phi} - Q_x = 0 \qquad (5)$$

About $p_r$ axis:

$$-N_{x\phi} r\,d\phi\,dx + N_{\phi x}\,dx\,r\,d\phi - 2\,M_{\phi x}\,dx\frac{d\phi}{2} = 0$$

or

$$N_{x\phi} - N_{\phi x} + \frac{1}{r} M_{\phi x} = 0 \tag{6}$$

Elimination of $Q_x$ and $Q_\phi$ by use of eq. (4) and (5) gives the following equations of equilibrium.

$$\frac{\partial}{\partial x} N_x + \frac{1}{r}\frac{\partial}{\partial \phi} N_{\phi x} + p_x = 0 \tag{7a}$$

$$\frac{1}{r}\frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} - \frac{1}{r}\frac{\partial M_{x\phi}}{\partial x} - \frac{1}{r^2}\frac{\partial M_\phi}{\partial \phi} + p_\phi = 0 \tag{7b}$$

$$\frac{1}{r} N_\phi + \frac{\partial^2 M_x}{\partial x^2} + \frac{1}{r}\frac{\partial^2 M_{\phi x}}{\partial x \partial \phi} + \frac{1}{r}\left[\frac{\partial^2 M_{x\phi}}{\partial \phi \partial x} + \frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial M_\phi}{\partial \phi}\right)\right] - p_r = 0 \tag{7c}$$

$$N_{x\phi} - N_{\phi x} + \frac{1}{r} M_{\phi x} = 0 \tag{7d}$$

At this point there are 4 equations and 8 unknowns. A study of the shell deformation will provide additional relationships.

In the analysis of deformations the following assumptions are made:

(1) All points lying on one normal to the middle surface before deformation do the same after deformation (i.e., neg-

lect deformation due to $Q_x$, $Q_\phi$ transverse shears).

(2) For all kinematic relations, the dist. Z of a point from the middle surface may be considered as unaffected by the deformation of theshell, but that for all considerations of the stress system, the stress $\sigma_z$ in the $z$ direction may be considered negligible compared to $\sigma_x$ and $\sigma_\phi$.

(3) All displacements are small, i.e., negligible compared with radii of curvature of the middle surface and their first derivatives are negligible compared to unity (linear assumption).

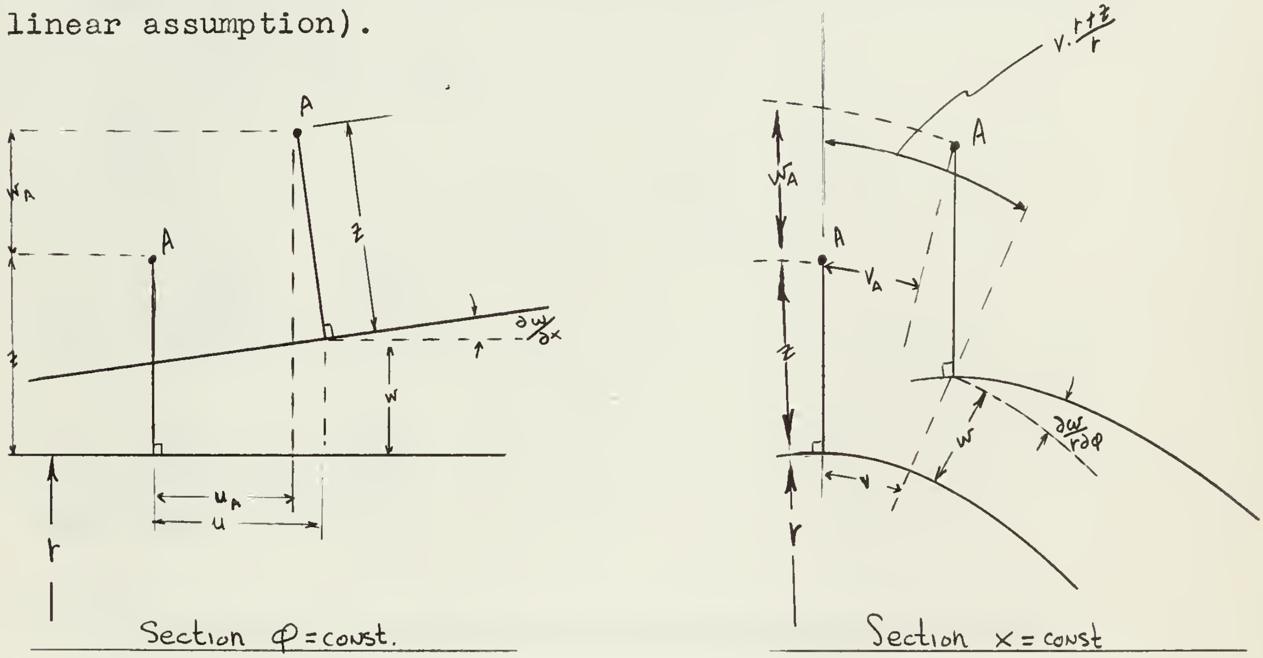



Fig. 4 Displacement of Point A

U, V, W refer to displacements in middle surface of shell in directions x, $\phi$, and z respectively.

$U_A$, $V_A$, $W_A$ refer to displacements of an arbitrary point A at

a point Z from the middle surface.

from Fig. 4 :

$$U_A = U - Z \frac{\partial W}{\partial x}$$

$$V_A = V\left(\frac{r + Z}{r}\right) - \frac{Z}{r}\frac{\partial W}{\partial \phi}$$

$$W_A = W \qquad \text{(assumption 2)}$$

The strains at point A can be expressed by the following relations from Flügge (6).

$$\epsilon_x = \frac{\partial u_A}{\partial x}$$

$$\epsilon_\phi = \frac{1}{r+z}\left(\frac{\partial v_A}{\partial \phi} + w_A\right)$$

$$\gamma_{x\phi} = \frac{\partial v_A}{\partial x} + \frac{1}{r+z}\frac{\partial u_A}{\partial \phi}$$

The strains may now be expressed in terms of middle surface displacement by using point A displacement relations.

$$\epsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_\phi \quad \frac{1}{r+z}\left\{\frac{\partial}{\partial \phi}\left[v(1+\frac{z}{r}) - \frac{z}{r}\frac{\partial w}{\partial \phi}\right] + w\right\}$$

$$\gamma_{x\phi} \quad \frac{\partial}{\partial x}\left[v\left(1+\frac{z}{r}\right) - \frac{z}{r}\frac{\partial w}{\partial \phi}\right] + \frac{1}{r+z}\left[\frac{\partial u}{\partial \phi} - z\frac{\partial^2 w}{\partial \phi \partial x}\right]$$

A simplification can be made at this point by considering the shell thickness to be very small in relation to the radius at any point. In otherwords $\frac{z}{r}$ is negligible compared to 1.
or

$$(1 + z/r) \approx 1$$

$$\frac{1}{r + z} \approx \frac{1}{r}$$

The strain relations now become

$$\epsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_\phi = \frac{1}{r} \left[ \frac{\partial}{\partial \phi} \left( v - \frac{z}{r} \frac{\partial w}{\partial \phi} \right) + w \right]$$

$$\gamma_{x\phi} = \frac{\partial}{\partial x} \left( v - \frac{z}{r} \frac{\partial w}{\partial \phi} \right) + \frac{1}{r} \left( \frac{\partial u}{\partial \phi} - z \frac{\partial^2 w}{\partial \phi \partial x} \right)$$

Carrying out the indicated operations and remembering that $r$ is a function of $\phi$ ($r = 2H\cos\phi$), we obtain the following relations when the notations $\frac{\partial}{\partial x} = (\ )'$ and $\frac{\partial}{\partial \phi} = (\dot{\ })$ are employed.

$$\epsilon_x = u' - z w'' \qquad (8a)$$

$$\epsilon_\phi = \frac{\dot{v}}{r} - \frac{z}{r^2} (\dot{w} \tan \phi + \ddot{w}) \quad \frac{w}{r} \qquad (8b)$$

$$\gamma_{x\phi} \quad v' \quad \frac{\dot{u}}{r} - 2 \frac{z}{r} \dot{w}' \qquad (8c)$$

The stress resultants of Fig. 3 are by definition

$$N_x = \int_{-t/2}^{t/2} \sigma_x (1 + z/r)\, dz \quad , \qquad N_\phi = \int_{-t/2}^{t/2} \sigma_\phi\, dz$$

$$N_{x\phi} = \int_{-t/2}^{t/2} \tau_{x\phi}(1 + z/r)\, dz \quad , \qquad N_{\phi x} = \int_{-t/2}^{t/2} \tau_{\phi x}\, dz$$

$$M_x = -\int_{-t/2}^{t/2} \sigma_x(1 + z/r)\, zdz \quad , \qquad M_\phi = -\int_{-t/2}^{t/2} \sigma_\phi zdz \qquad (9)$$

$$M_{x\phi} = -\int_{-t/2}^{t/2} \tau_{x\phi}(1 + z/r)\, zdz \quad , \qquad M_{\phi x} = -\int_{-t/2}^{t/2} \tau_{\phi x} zdz$$

The terms $(1 + z/r)$ goes to 1 however by the previous assumption of negligible $z/r$ ratio.

These stress resultants can now be found with the aid of the strain relations ( 8 ).

since

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\,\epsilon_\phi)$$

$$\sigma_\phi = \frac{E}{1-\nu^2}(\epsilon_\phi + \nu\epsilon_x) \qquad (10)$$

$$\tau_{x\phi} = \frac{E}{2(1+\nu)}\,\gamma_{x\phi}$$

After substitution of ( 10 ) and ( 8 ) in ( 9 ) the following stress resultant relations are obtained where

$D = \dfrac{Et}{1-\nu^2}$ is the extensional rigidity and

$K = \dfrac{Et^3}{12(1-\nu^2)}$ is the flexural rigidity or bending stiffness.

$$N_x = \frac{D}{r}(ru' + \nu\dot{v} + \nu w)$$

$$N_\phi = \frac{D}{r}(\dot{v} + w + \nu ru') \qquad (11)$$

$$N_{x\phi} = N_{\phi x} = D\,\frac{(1-\nu)}{2r}(\dot{u} + rv')$$

$$M_x = \frac{K}{r^2} \left( r^2 w'' + \nu \ddot{w} + \nu \tan \phi \, \dot{w} \right)$$

$$M_\phi = \frac{K}{r^2} \left( \ddot{w} + \tan \phi \, \dot{w} + \nu r^2 w'' \right) \qquad \text{(11 con't)}$$

$$M_{x\phi} = M_{\phi x} = K \left( \frac{1-\nu}{r} \right) \dot{w}'$$

These relations represent the stress resultants of an isotropic cylindrical shell with variable curvature $r = 2H\cos\phi$ . (Note that the shell thickness has not been fixed yet but can still be a function of $\phi$). These relations exist in the unstiffened ships hull.

We consider now the gridwork of frames and longitudinal stringers only in Figure 5 .

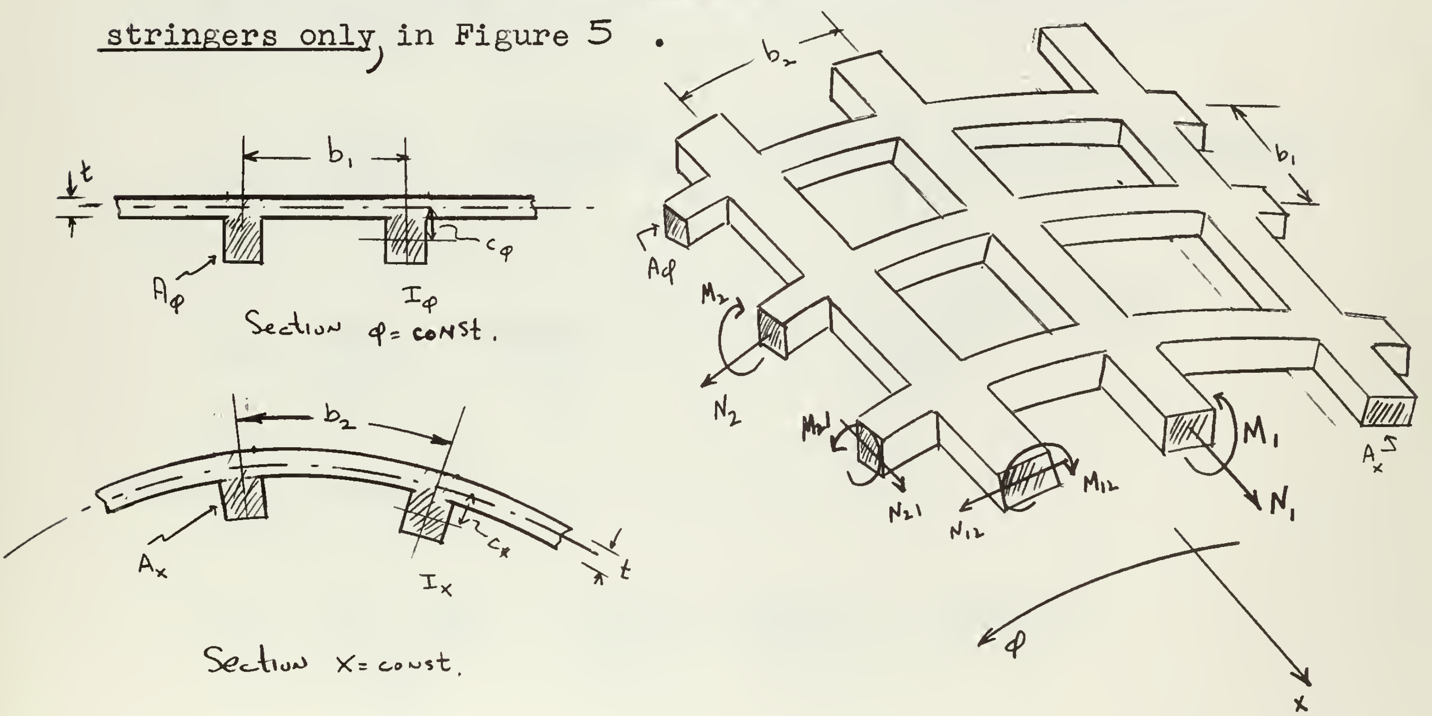



Fig. 5 Element of a Gridwork Shell

$$N_2 = EA_\phi \epsilon_\phi \qquad \text{where } \epsilon_\phi = \text{strain at centroid}$$

then $N_\phi = \dfrac{N_2}{b_1} = \dfrac{EA_\phi \epsilon_\phi}{b_1}$

Using the strain relations of (eq.8) with $z = C_\phi$

$$N_\phi = \frac{EA_\phi}{b_1}\left[\frac{\dot{V}}{r} - \frac{C_\phi}{r^2}\left(\ddot{w} + \tan\phi\,\dot{w}\right) + \frac{w}{r}\right]$$

similarly

$$N_x = \frac{EA_x \epsilon_x}{b_2} = \frac{EA_x}{b_2}\left(u' - C_x w''\right)$$

$$M_\phi = \frac{M_2 - N_2 C_\phi}{b_1}$$

$$= \frac{EI_\phi \ddot{w}}{b_1 r^2} - \frac{EA_\phi C_\phi \epsilon_\phi}{b_1}, \qquad \text{where } \frac{\ddot{w}}{r^2} = \text{change in curvature}$$

$$M_\phi = \frac{EI_\phi \ddot{w}}{b_1 r^2} - \frac{EA_\phi C_\phi}{b_1}\left(\dot{v}/r + w/r\right) + \frac{EA_\phi C_\phi^2\left(\ddot{w} + \tan\phi\,\dot{w}\right)}{b_1 r^2}$$

similarly for $M_x$,

$$M_x = \frac{M_1 - N_1 C_x}{b_2} = \frac{EI_x w''}{b_2} - \frac{EA_x C_x u'}{b_2} + \frac{EA_x C_x^2 w''}{b_2}$$

$$M_{x\phi} = \frac{M_{12}}{b_2} \quad , \qquad M_{\phi x} = \frac{M_{21}}{b_1}$$

twist of a stringer $= \theta_1 = \dfrac{M_{12}}{GJ_x} = b_1 \dfrac{\dot{w}'}{r}$

$$\theta_1 b_1 = \frac{M_{12} b_1}{GJ_x} = \frac{M_{x\phi} b_1 b_2}{GJ_x} = \frac{b_1 \dot{w}'}{r}$$

$$M_{x\phi} = \frac{GJ_x}{b_2} \frac{\dot{w}'}{r}$$

similarly, $M_{\phi x} = \dfrac{GJ_\phi}{b_1} \dfrac{\dot{w}'}{r}$

note that $\dfrac{M_{\phi x}}{M_{x\phi}} = \dfrac{J_\phi}{J_x} \cdot \dfrac{b_2}{b_1}$

Summarizing the gridwork <u>stress resultants of a variable curvature cylindrical shell form $(r = 2H\cos\phi)$</u>. —

$$N_x = \frac{EA_x}{b_2} (u' - C_x w'')$$

$$N_\phi = \frac{EA_\phi}{b_1} \left[ \frac{\dot{v}}{r} - \frac{C_\phi}{r^2} (\ddot{w} + \tan\phi \dot{w}) + w/r \right] \tag{12}$$

$$M_x = \frac{EI_x w''}{b_2} - \frac{EA_x C_x u'}{b_2} + \frac{EA_x C_x^2 w''}{b_2}$$

$$M_\phi = \frac{EI_\phi \ddot{w}}{b_1 r^2} - \frac{EA_\phi C_\phi}{b_1}(\dot{v}/r + w/r) + \frac{EA_\phi C_\phi^2}{b_1 r^2}(\ddot{w} + \tan\phi\dot{w})$$

$$M_{x\phi} = \frac{GJ_x}{b_2}\frac{\dot{w}'}{r} \qquad (12\text{ con't})$$

$$M_{\phi x} = \frac{GJ_\phi}{b_1}\frac{\dot{w}'}{r}$$

NOTE: $N_{x\phi}$ and $N_{\phi x}$ could also be determined but it will be seen shortly that they will not be necessary.

These relations exist in the ships hull stiffners.

Superposing the stress resultants of the isotropic shell with those of the gridwork gives final stress resultants of the stiffened variable curvature shell (or the stiffened ships hull).

$$N_x = D_x u' + \frac{D_\nu}{r}(\dot{v} + w) - S_x w''$$

$$N_\phi = \frac{D_\phi(\dot{v} + w)}{r} + D_\nu u' - \frac{S_\phi}{r^2}(\ddot{w} + \tan\phi\dot{w}) \qquad (13)$$

$$M_x = K_x w'' + \frac{K_\nu}{r^2}(\ddot{w} + \tan\phi\dot{w}) - S_x u'$$

$$M_\phi = K_\phi\frac{\ddot{w}}{r^2} + k_\nu\left(w'' + \frac{\tan\phi\dot{w}}{\nu r^2}\right) - \frac{S_\phi}{r}(\dot{v} + w)$$

$$N_{x\phi} = \frac{D_{x\phi}}{r}(\dot{u} + rv')$$

$$N_{\phi x} = \frac{D_{x\phi}}{r}(\dot{u} + rv') + K_{\phi x}\frac{\dot{w}'}{r^2} \qquad (13\text{ con't})$$

$$M_{\phi x} = \frac{K_{\phi x}}{r}\dot{w}'$$

$$M_{x\phi} = \frac{K_{x\phi}}{r}\dot{w}'$$

Where the orthotropic rigidity coeffecients are defined as

$$D_x = \frac{Et}{1-\nu^2} + \frac{EA_x}{b_2} \qquad D_\phi = \frac{Et}{1-\nu^2} + \frac{EA_\phi}{b_1}$$

$$D_\nu = \frac{Et\nu}{1-\nu^2} \qquad D_{x\phi} = \frac{Et}{2(1+\nu)} \qquad (14)$$

$$K_x = \frac{Et^3}{12(1-\nu^2)} + \frac{E(I_x + A_x C_x^2)}{b_2}, \quad K_\phi = \frac{Et^3}{12(1-\nu^2)} + \frac{E(I_\phi \quad A_\phi C_\phi^2)}{b_1}$$

$$K_\nu = \frac{Et^3\nu}{12(1-\nu^2)}, \qquad S_x = \frac{EA_xC_x}{b_2}, \qquad S_\phi = \frac{EA_\phi C_\phi}{b_1}$$

$$K_{x\phi} = \frac{Et^3}{12(1+\nu)} + \frac{GJ_x}{b_2}, \qquad K_{\phi x} = \frac{Et^3}{12(1+\nu)} + \frac{GJ_\phi}{b_1}$$

Note that the $N_{x\phi}$ and $N_{\phi x}$ represent shear carried by the shell alone. The gridwork can only carry this shear by a bending of its stiffeners, a bending which becomes impossible when the shell is joined to the stiffeners.

Introducing, now, these new stress resultants for the stiffened ships hull, into the equations of equilibrium (eqs. 7), one obtains the following differential equations.

$$D_x u'' + \frac{D_\nu}{r}(\dot{v}' + w') - S_x w''' + \frac{1}{r}\frac{\partial}{\partial\phi}\left[D_{x\phi}\left(\frac{\dot{u}}{r} + v'\right) + K_{\phi x}\frac{\dot{w}'}{r^2}\right] + p_x = 0$$

$$\frac{1}{r}\frac{\partial}{\partial\phi}\left[\frac{D_\phi}{r}(\dot{v} + w) + D_\nu u' - \frac{S_\phi}{r^2}(\ddot{w} + \tan\phi\dot{w})\right] + \frac{D_{x\phi}}{r}(\dot{u}' + rv'') - \frac{K_{x\phi}}{r^2}\dot{w}'' - \frac{1}{r^2}\left\{\frac{\partial}{\partial\phi}\left[\frac{K_\phi\ddot{w}}{r^2} + K_\nu w'' + K_\nu\frac{\tan\phi\dot{w}}{\nu r^2} - \frac{S_\phi}{r}(\dot{v} + w)\right]\right\} + p_\phi = 0$$

$$\frac{D_\phi}{r^2}(\dot{v} + w) + \frac{D_\nu}{r}u' - \frac{S_\phi}{r^3}(\ddot{w} + \tan\phi\dot{w}) + K_x w'''' + \frac{K_\nu}{r^2}(\ddot{w}'' + \tan\phi\dot{w}'') - S_x u''' + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{K_{\phi x}}{r}\dot{w}''\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left\{\frac{K_{x\phi}}{r}\dot{w}'' + \frac{1}{r}\frac{\partial}{\partial\phi}\left[\frac{K_\phi\ddot{w}}{r^2} + K_\nu\left(w'' + \frac{\tan\phi\dot{w}}{\nu r^2}\right) - \frac{S_\phi}{r}(\dot{v} + w)\right]\right\} - p_r = 0$$

After carrying out the indicated operations, rearranging terms and using partial operator notation as follows,

$$d_1 = \frac{\partial}{\partial x} = (\ )' \quad , \quad d_2 = \frac{\partial}{\partial \phi} = (\dot{\ })$$

and $$d_1^n = \frac{\partial^n}{\partial x^n} \quad , \quad d_2^n = \frac{\partial^n}{\partial \phi^n}$$

the differential equations assume the form,

$$\left[C_1 r^3 d_1^2 + C_2 r d^2{}_2 + C_2 r \tan\phi d_2\right] u + \left[C_3 r^2 d_1 d_2\right] v \tag{15a}$$

$$\left[C_4 r^2 d_1 - C_5 r^3 d_1{}^3 + C_6 d_1 d_2{}^2 + 2C_6 \tan\phi\, d_1 d_2\right] w = -r^3 p_x$$

$$\left[C_3 r^3 d_1 d_2\right] u + \left[C_7 r^2 d_2{}^2 + C_8 r d_2{}^2 + C_2 r^4 d_1{}^2 + C_7 r^2 \tan\phi\, d_2\right.$$

$$\left. + C_8 r \tan\phi\, d_2\right] v + \left[-C_9 d^3{}_2 - C_8 r d_2{}^3 - C_{10} r^2 d_1^2 d_2 + C_7 r^2 d_2\right.$$

$$-3C_8 r \tan^2\phi\, d_2 - C_{11} d_2 - 3C_{11} \tan^2\phi\, d_2 - C_8 r \tan\phi\, d_2{}^2 \tag{15b}$$

$$\left. -2C_9 \tan\phi\, d_2{}^2 - C_{11} \tan\phi\, d_2{}^2 + C_7 r^2 \tan\phi + C_8 r \tan\phi\right] w = -r^4 p_\phi$$

$$\left[-C_5 r^4 d_1{}^3 + C_4 r^3 d_1\right] u + \left[-C_8 r d_2{}^3 - 3C_8 r \tan\phi\, d_2{}^2\right.$$

$$\left. + C_7 r^2 d_2 - 3C_8 r \tan^2\phi\, d_2 - C_8 r d_2\right] v \quad +$$

$$\left[C_{12} r^4 d_1{}^4 + C_{13} r^2 d_1{}^2 d_2{}^2 + C_9 d_2{}^4 + C_{13} r^2 \tan\phi\, d_1{}^2 d_2\right.$$

$$+ C_{14} \tan\phi\, d_2{}^3 - 2C_8 r d_2{}^2 + 8C_9 \tan^2\phi\, d_2{}^2 + 2C_9 d_2{}^2 \tag{15c}$$

$$+ 2C_{11} d_2{}^2 + 7C_{11} \tan^2\phi\, d_2{}^2 - 4C_8 r \tan\phi\, d_2 + 9C_{11} \tan\phi\, d_2$$

$$\left. + 15C_{11} \tan^3\phi\, d_2 + C_7 r^2 - 3C_8 r \tan^2\phi - C_8 r\right] w = r^4 p_r$$

The constant part of the coefficients are identified as

$$C_1 = D_x \qquad C_8 = S_\phi$$

$$C_2 = D_{x\phi} \qquad C_9 = K_\phi$$

$$C_3 = D_\nu + D_{x\phi} \qquad C_{10} = K_\nu + K_{x\phi}$$

$$C_4 = D_\nu \qquad C_{11} = \frac{K_\nu}{\nu}$$

$$C_5 = S_x \qquad C_{12} = K_x$$

$$C_6 = K_{\phi x} \qquad C_{13} = K_{x\phi} + K_{\phi x} + 2K_\nu$$

$$C_7 = D_\phi \qquad C_{14} = 5K_\phi + \frac{K_\nu}{\nu}$$

Equations (15, a, b, c) comprise a set of 3 partial differential equations in 3 unknowns, (u,v,w). Their solution as a function

of radius and rigidity coefficients will completely define the structural behavior of a stiffened cylindrical shell (of cycloid cross section) for given loading and boundary conditions. The rigidity coefficients were made constant as a necessary simplification. It will be seen later, however, that many are in actual practice essentially constant over considerable portions of the shell section. Additional comments concerning these differential equations will be delayed until the Discussion of Results section.

## Membrane Analysis

Since the solution of the developed differential equations is not readily available, one must approach the shell problem with other and more immediately useful tools of analysis. It must be stated at this point, however, that the analytical approach by means of the differential equations of the problem is the only means by which a complete and relatively exact solution may be obtained. It is unfortunate that their complex nature is currently limiting their use by shell designers. The alternate methods employed to obtain solutions include the beam method, iterative calculation techniques, membrane theory, and other unique approximation approaches. Lundgren (7) describes rather completely the various methods of shell analysis currently in use.

The applicability or the relative desireability of the alternate methods is determined basically by the shell dimensions. The ratio of the shell length to the shell span (or beam), the L/B ratio, normally will govern the technique. The

application of membrane theory (i.e., the consideration of direct stresses only; the support of the load by tension, compression or shear forces only) is considered good for short shells or shells whose L/B $\leq$ 1. Lundgren (7) states that the membrane condition of stress is a very good approximation for the short shell since the elastic support between end sections (or bulkheads) is quite stiff and loading is balanced by $N_x$ , $N_\phi$ , and $N_{x\phi}$ stresses. The ASCE Manual of Engineering Practice No. 31 recommends a superposition of membrane stresses and those created by boundary conditions. However, on the matter of boundary conditions, the manual also states that for a L/B $<$ .83 the transverse (Bulkhead) end effects damp rapidly and the effects of the longitudinal edges are felt only in a small area.

The previous discussion indicates that a membrane analysis of a ships bottom section between bulkheads is a reasonable approach when one realizes that the L/B range for the ships or ship type under consideration is as indicated in Table 1.

| | DE 1033 | DDG 952 | DLG 9 | DLG(N) | CG(N) | CA 134 |
|---|---|---|---|---|---|---|
| L/B | 0.775 | 0.85 | 0.81 | 0.965 | 0.56 | 0.56 |

L = length between bulkheads (IN $\otimes$ region)
B = beam at waterline

TABLE 1 Length to Span Ratio of Ships Considered

The membrane stresses (lb/ft) in the shell element can be obtained by setting the shell moments equal to zero in the equations of equilibrium (eqs. 7).

$$N_\phi = p_r r$$

$$\frac{\partial N_{x\phi}}{\partial x} = -p_\phi - \frac{1}{r}\frac{\partial N_\phi}{\partial \phi}$$

$$\frac{\partial N_x}{\partial x} = -p_x - \frac{1}{r}\frac{\partial N_{x\phi}}{\partial \phi}$$

integrating the latter two of these,

$$N_{x\phi} = -\int\left(p_\phi + \frac{1}{r}\frac{\partial N_\phi}{\partial \phi}\right) dx + f_1(\phi)$$

$$N_x = -\int\left(p_x + \frac{1}{r}\frac{\partial N_{x\phi}}{\partial \phi}\right) dx + f_2(\phi)$$

for $p_x = 0$ and $p_\phi$ and $p_r$ independent of x. (These are correct conditions for ships hull loading. The longitudinal stress resulting from the hull girder action and determined from the longitudinal strength calculation will be accounted for shortly but will not be considered as a $p_x$ loading at this point).

$$N_{x\phi} = -\left(p_\phi + \frac{1}{r}\frac{d N_\phi}{d\phi}\right) x + f_1(\phi) = -xF_{(\phi)} + f_1(\phi)$$

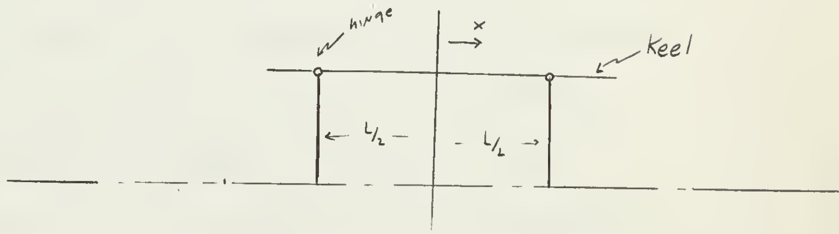
Where $F(\phi) = \left(p_\varphi + \frac{1}{r}\frac{d N_\varphi}{d\varphi}\right)$

$$N_x = \frac{x^2}{2r}\frac{dF(\phi)}{d\phi} - \frac{x}{r}\frac{df_1(\phi)}{d\phi} + f_2(\phi)$$

$F(\phi)$ is determined by loading and shell shape (curvature) but $f_1(\phi)$ and $f_2(\phi)$ will be determined by boundary conditions at the shell ends (bulkheads).

Consider first the case of <u>simple support end (bulkhead) conditions</u>.

At $X = \pm L/2$, $N_x = 0$




then,

Fig. 6 Shell on Simple Supports

$$0 = \frac{L^2}{8r}\frac{dF(\phi)}{d\phi} - \frac{L}{2r}\frac{df_1(\phi)}{d\phi} + f_2(\phi)$$

$$0 = \frac{L^2}{8r}\frac{dF(\phi)}{d\phi} + \frac{L}{2r}\frac{df_1(\phi)}{d\phi} + f_2(\phi)$$

which gives,

$$f_1(\phi) = 0 \qquad f_2(\phi) = -\frac{L^2}{8r}\frac{dF(\phi)}{d\phi}$$

and therefore,

$$N_{x\phi} = -xF(\phi)$$

$$N_x = -\frac{1}{8r}(L^2 - 4x^2)\frac{dF(\phi)}{d\phi}$$

Now consider shell end or bulkhead conditions for the shell continuous over the supports and assume 100% clamping. As an aid in determining the stress $N_x$ over the supports, one can make use of the relationship given in ASCE Manual No. 31: the ratio of the stress (in a shell over continuous supports) at the point of support to the stress in simply supported shell at the midspan is equal to ratio of the moment in the continuous shell at the support to the moment in the simply supported shell at the midspan.

or

$$\frac{\sigma_E}{\sigma_0} = \frac{M_E}{M_0}$$

and, for a strip of shell,

$$M_E = -1/12\ wL^2$$

$$M_0 = 1/8\ wL^2$$

$$\frac{N_{xE}}{N_{x0}} = -2/3$$

$$N_{xE} = -2/3\ N_{x0}$$

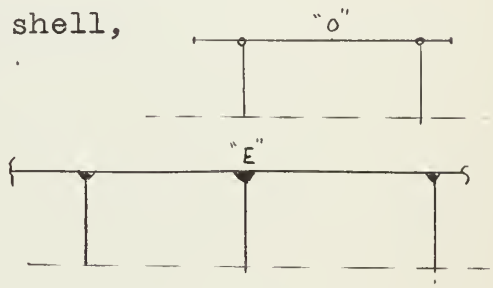



Fig. 7 Comparison of Continuous & Simple Supports for Shell

and $N_{x0}$ was found previously to be $-\dfrac{L^2}{8r}\dfrac{dF(\phi)}{d\phi}$

$N_x$ at support $(X = \pm L/2)$ is $+\dfrac{L^2}{12r}\dfrac{dF(\phi)}{d\phi}$ for a shell continuous over supports.

At $X = \pm L/2$

$$N_x = +\frac{L^2}{12r}\frac{dF(\phi)}{d\phi}$$

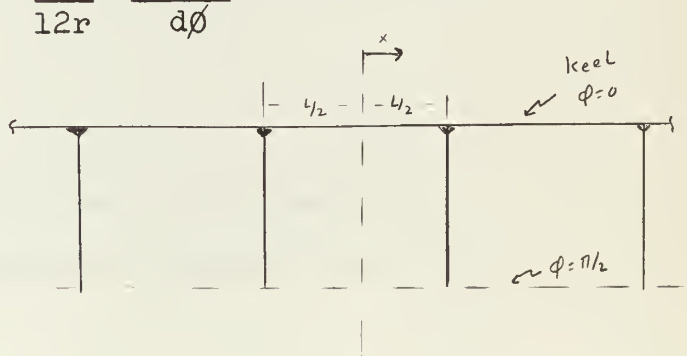



Fig. 8 Shell Continuous over Supports

or

$$\frac{L^2}{12r}\frac{dF(\phi)}{d\phi} = \frac{L^2}{8r}\frac{dF(\phi)}{d\phi} - \frac{L}{2r}\frac{df_1(\phi)}{d\phi} + f_2(\phi)$$

$$\frac{L^2}{12r}\frac{dF(\phi)}{d\phi} = \frac{L^2}{8r}\frac{dF(\phi)}{d\phi} + \frac{L}{2r}\frac{df_1(\phi)}{d\phi} + f_2(\phi)$$

which gives,

$$f_1(\phi) = 0$$

$$f_2(\phi) = -\frac{L^2}{24r}\frac{dF(\phi)}{d\phi}$$

then,

$$N_{x\phi} = -xF(\phi)$$

$$N_x = -\frac{(L^2 - 12x^2)}{24\,r}\frac{dF(\phi)}{d\phi} \qquad (16)$$

$$N_\phi = p_r\,r$$

where,

$$F(\phi) = (p_\phi + \frac{1}{r}\frac{d\,N_\phi}{d\,\phi})$$

Equation 16 represent the general form of the membrane stresses in the ships bottom section of radus of curvature r.

These stresses will now be developed in a more final

form by considering the hydrostatic loading and the cycloidal cross section cylinder.

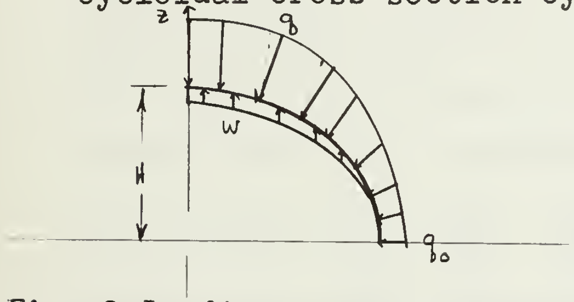



Fig. 9 Loading on Ships Hull

Let $q_0 = \gamma\, h_0$ where $h_0 =$ an arbitrary design head to account for wave forces on freeboard surface of hull.

$z = H/2\,(1+\cos 2\phi)$ from appendix (A)

$w =$ internal loading of shell due to machinery weights ($lb/ft^2$).

$$q = q_o + \gamma z$$
$$q = \gamma (h_o + z)$$
$$q = \gamma\left[h_o + H/2\,(1+\cos 2\phi)\right]$$
$$p_r = -(q - w\cos\phi)$$

$$\begin{cases} p_r = -\gamma\left[h_0 + H/2(1+\cos 2\phi)\right] + w\cos\phi \\ p_\phi = -w\sin\phi \\ p_x = 0 \end{cases}$$

$$N_\phi = \left\{-\gamma\left[h_0 + H/2(1+\cos 2\phi)\right] + w\cos\phi\right\} 2H\cos\phi$$

$$N_\phi = -\gamma\left[2H\,h_0\cos\phi + 2H^2\cos^3\phi\right] + 2wH\cos^2\phi \qquad (17a)$$

and since $F(\phi) = p_\phi + \frac{1}{r}\,\frac{dN_\phi}{d\phi}$

$$F(\phi) = \gamma\left[h_0\tan\phi + \frac{3H}{2}\sin 2\phi\right] - 3w\sin\phi$$

$N_x$ and $N_{x\phi}$ are then found by applying equation (16).

$$N_{x\phi} = -x\left[\gamma(h_0\tan\phi + 3/2H\,\sin 2\phi) - 3w\sin\phi\right] \qquad (17b)$$

$$N_x \quad \frac{L^2 - 12x^2}{48H}\left[3w - \gamma\left(h_0\sec^3\phi + \frac{3H\cos 2\phi}{\cos\phi}\right)\right] \qquad (17c)$$

Equation (17) are the membrane stresses for the ships bottom section. With the aid of these relations, the necessary shell thickness (that is, the equivalent shell thickness for a hull with no stiffeners) can be found as it varies around the girth. From this equivalent thickness, extensional rigidity coefficients ($\frac{D_x}{E}$, $\frac{D_\phi}{E}$, $\frac{D_{x\phi}}{E}$, $\frac{D_\nu}{E}$) may be computed. For the single equivalent thickness derived there results the fact that $\frac{D_x}{E} = \frac{D_\phi}{E}$ since they both equal $\frac{t_e}{1-\nu^2}$ or $1.1t_e$ see appendix D. However, in practice these coefficients are very nearly equal as can be seen from Figure 26.

$$\sigma_x = \frac{N_x}{12\ t_e} + \sigma_1$$

$$\sigma_\phi = \frac{N_\phi}{12\ t_e} \qquad (18)$$

$$\tau_{x\phi} = \frac{N_{x\phi}}{12\ t_e}$$

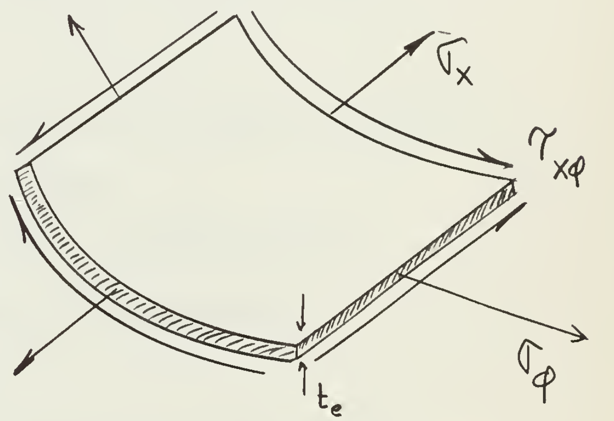



Figure 10 Direct Stresses on Equivalent Shell Element

Where $\sigma_1$ = the hull girder longitudinal bending stress as determined from the longitudinal strength calculation.

Using the maximum shear failure theory and in particular the Tresca version,

$$\frac{\sigma_y}{2} = \tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_\phi}{2}\right)^2 + \tau_{x\phi}^2} \qquad \text{or} \qquad (19)$$

$$\sigma_y^2 = \sigma_x^2 - 2\sigma_x\sigma_\phi + \sigma_\phi^2 + 4\tau_{x\phi}^2$$

Substituting equation (18) in equation (19) and solving for $t_e$ gives

$$t_e = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad (20)$$

where $A = 144\,(\sigma_1^{\,2} - \sigma_y^{\,2})$

$$B = 24\,\sigma_1(N_x - N_\phi)$$

$$C = N_x^{\,2} - 2N_xN_\phi + N^2_\phi + 4N_{x\,\phi}^{\,2}$$

It now becomes necessary to make a judgement as to the treatment of $\sigma_1$ and $\sigma_y$.

$\sigma_y$ will be assumed as 33,000 psi and a safety factor of 1.65 will reduce its value to 20,000 psi.

$\sigma_1$ will be approximated by Tobins relation that

$\sigma_{1\ max} = \sqrt[3]{L_s}$ where $L_s$ = length of ship in ft.

$\sigma_1$ = Ton/in$^2$

If it is desired to have $\sigma_1$ as a function of ship depth (or girth), with $\sigma_1 = \sigma_{1\ max}$ at the keel ($\phi = 0$) and $\sigma_1 = \frac{\sigma_{1\ max}}{4}$ at the waterline or N.A. ($\phi = \pi/2$) (to allow for the ship being in the heeled condition),

then $$\sigma_1 = \frac{\sqrt[3]{L_s}}{8}\,(5 + 3\cos 2\phi) \qquad (21)$$

The membrane analysis has now been sufficiently developed to make a test of its applicability to specific examples. For a given ship (or design), the equivalent shell thickness variation can be found by means of equation (20) which requires a determination of membrane stresses equation (17) and hull girder bending stress estimate equation (21). The extensional rigidity coefficients are then readily calculated since the equivalent shell has no stiffeners and $\frac{D_x}{E} \cong \frac{D_\phi}{E} = \frac{t_e}{1-\nu^2} = 1.1 t_e$

The calculated values may then be compared to the actual values on the ship selected for study. It should be noted that in the calculation of the membrane stresses, the longitudinal section ($X =$ constant) selected for stress determination was the section at the bulkhead. This was a logical design decision since a close examination of equation (20) (or a few comparative calculations) will indicate that analysis of this section results in the greatest $t_e$ for any given $\phi$ value along the shell length. In addition, the value of w or internal loading was assumed equal to zero. Not only will this be the more severe loading condition, but it is also the one which current design (series of flat panel analyses) employs. This assumption will assist in making the comparison more valid. The retention of this quantity in the general theory, however, is recommended. Additional comments on this subject will be made in the discussion section.

# III. RESULTS

The analysis of the orthotropic shell by superposition of the isotropic shell and a stiffener gridwork (with all effects of bending and torsional moments in the shell considered) resulted in a set of 3 partial differential equations in 3 unknowns (u, v, w). They are equations (15) on page 22. Due to their lengthy nature they will not be repeated here. The coefficients in the partial differential equations are variable in that they contain the radius of curvature in powers up to the fourth. Equation (15c) is of the fourth order while 15a & b are third order. Equation (15a) could be made homogeneous by setting $p_x = 0$ but equations 15b,c must remain non-homogeneous. All three partial differential equations are linear.

The membrane analysis produced more tangible results in the form of expressions for the membrane stresses as a function of: girth angle ($\phi$), position along shell length (X), and ship characteristics such as ship length ($L_s$), draft (H), compartment internal loading (w), distance between bulkheads (L), and a design head on freeboard plating ($h_0$).

Membrane stresses in ships bottom section.*

$$N_x = \frac{L^2 - 12x^2}{48H}\left[3w - \gamma\left(h_0 \sec^3\phi + 3H \frac{\cos 2\phi}{\cos\phi}\right)\right]$$

$$N_\phi = -\gamma(2H\, h_0 \cos\phi + 2H^2\cos^3\phi) + 2w\, H\cos^2\phi$$

$$N_{x\phi} = -x\left[\gamma\, (h_0\tan\phi + 3/2\, H \sin 2\phi) - 3w \sin\phi\right]$$

* A bottom section constructed with no stiffeners but an equivalent thickness ($t_e$) to produce the same extensional rigidity as one with stiffeners.

Figure 11 illustrates the membrane stress distribution along the length of the shell (a section $\phi =$ const).

Figure 12 represents the circumferential distribution of membrane stresses in DE 1033 and DD 931 hulls. (a section x const).

The derived expression for the shell equivalent thickness variation utilizes the membrane stresses along with the hull girder bending stress ($\sigma_1$) and a yield or allowable design stress ($\sigma_y$).

$$t_e = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad \text{inches}$$

where

$$A = 144\,(\sigma_1^{\,2} - \sigma_y^{\,2}) \qquad \sigma_1,\ \sigma_y,\ = \text{psi}$$

$$B = 24\,\sigma_1\,(N_x - N_\phi) \qquad \text{membrane stresses} = \text{lb/ft}$$

$$C = N_x^{\,2} - 2N_xN_\phi + N^2_\phi + 4N^2_{x\phi}$$

The extensional rigidity coefficients are then computed by

$$\frac{D_x}{E} \cong \frac{D_\phi}{E} = \frac{t_e}{1-\nu^2} = 1.1t_e \qquad \text{for } \nu = 0.3$$

$$D_{x\phi} = \frac{t_e}{2(1+\nu)} = .38t_e$$

$$D_\nu = \frac{t_e}{1-\nu^2} = .33t_e$$

Using the DE 1033 and DD 931 hulls, these coefficients have been calculated from the developed expressions above and compared to those calculated from the actual hull structure. The results are plotted in Figures 14, 15, 16, 17.

Figure 11 Longitudinal Distribution of Direct Stresses in Ships Hull

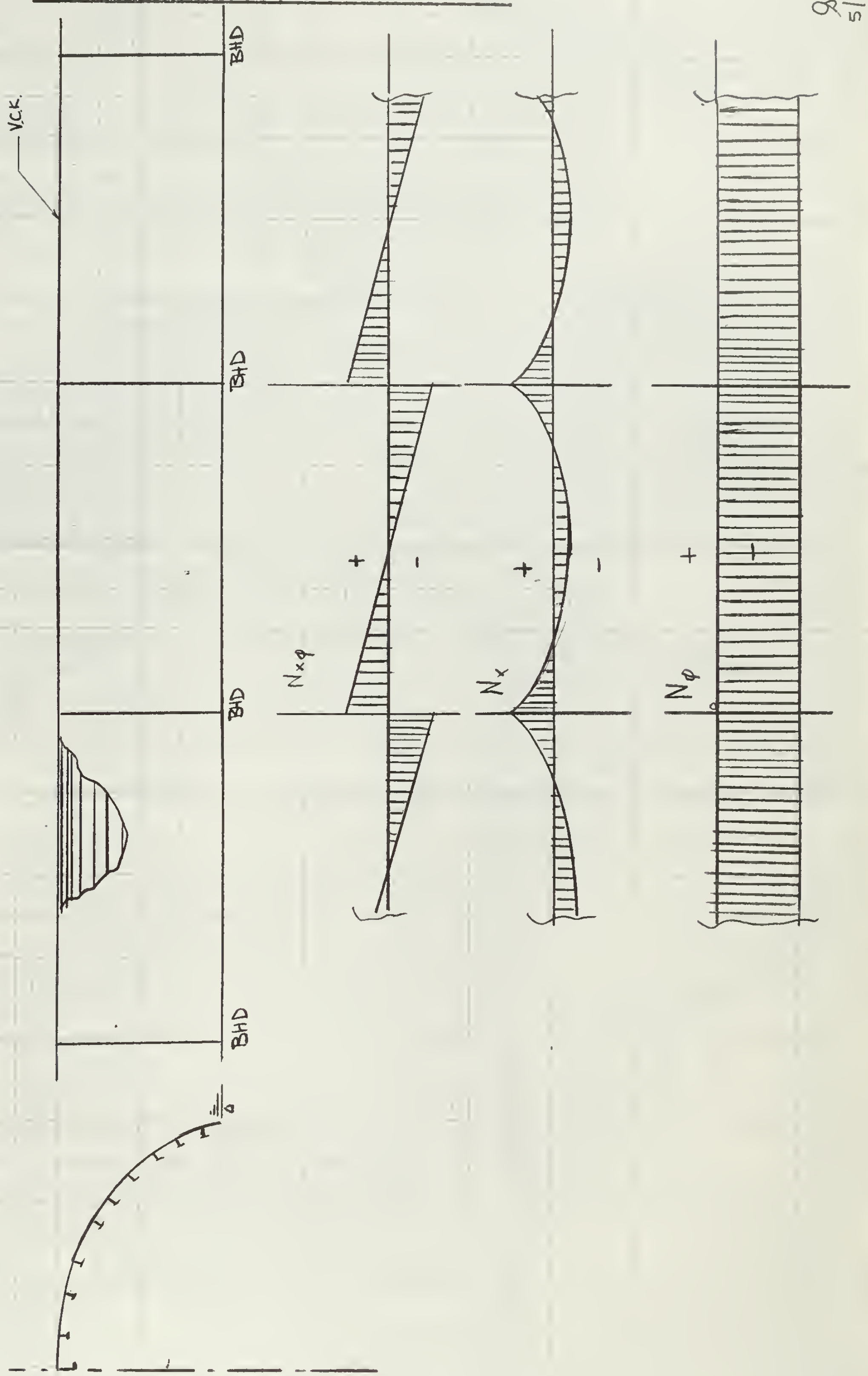
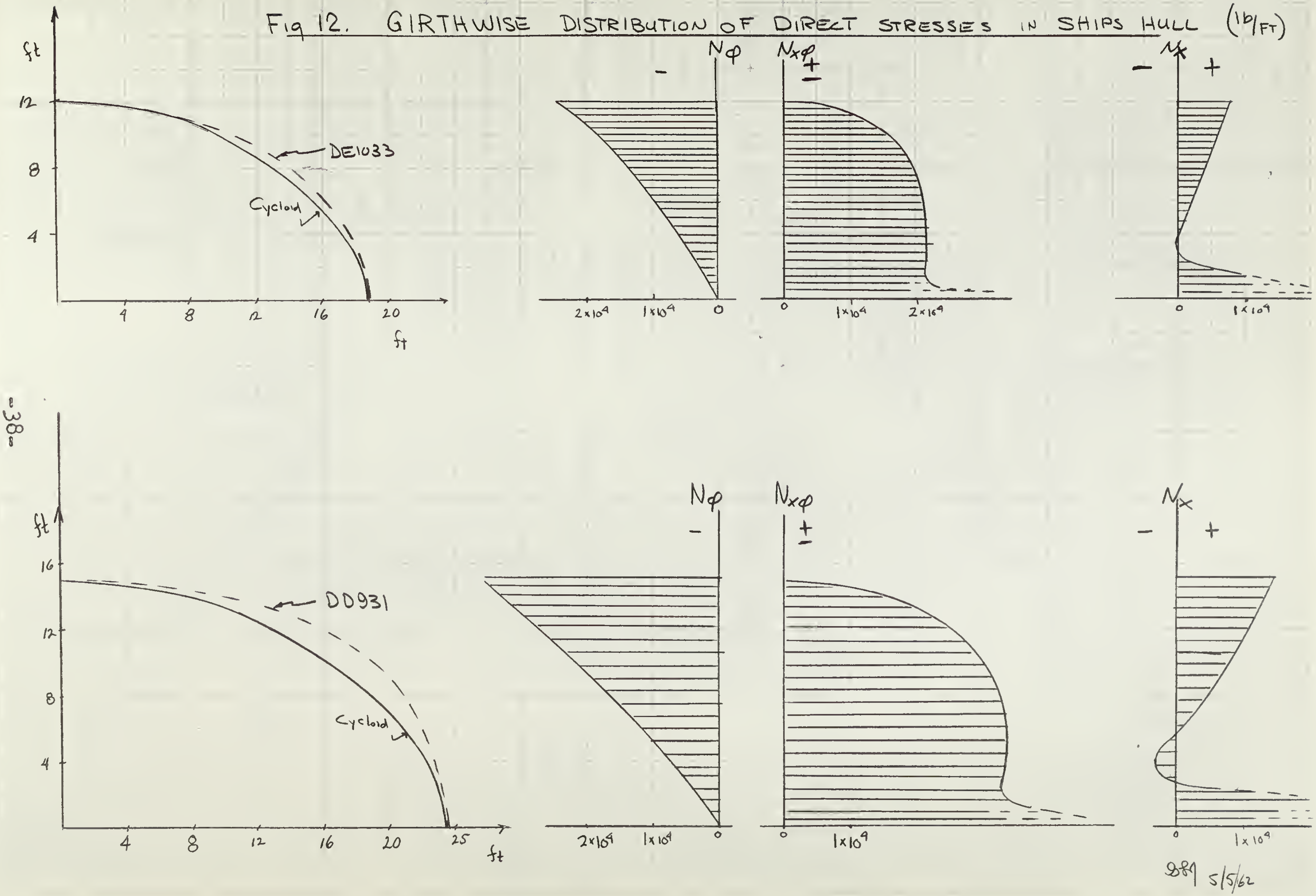


Fig 12. GIRTHWISE DISTRIBUTION OF DIRECT STRESSES IN SHIPS HULL (lb/FT)



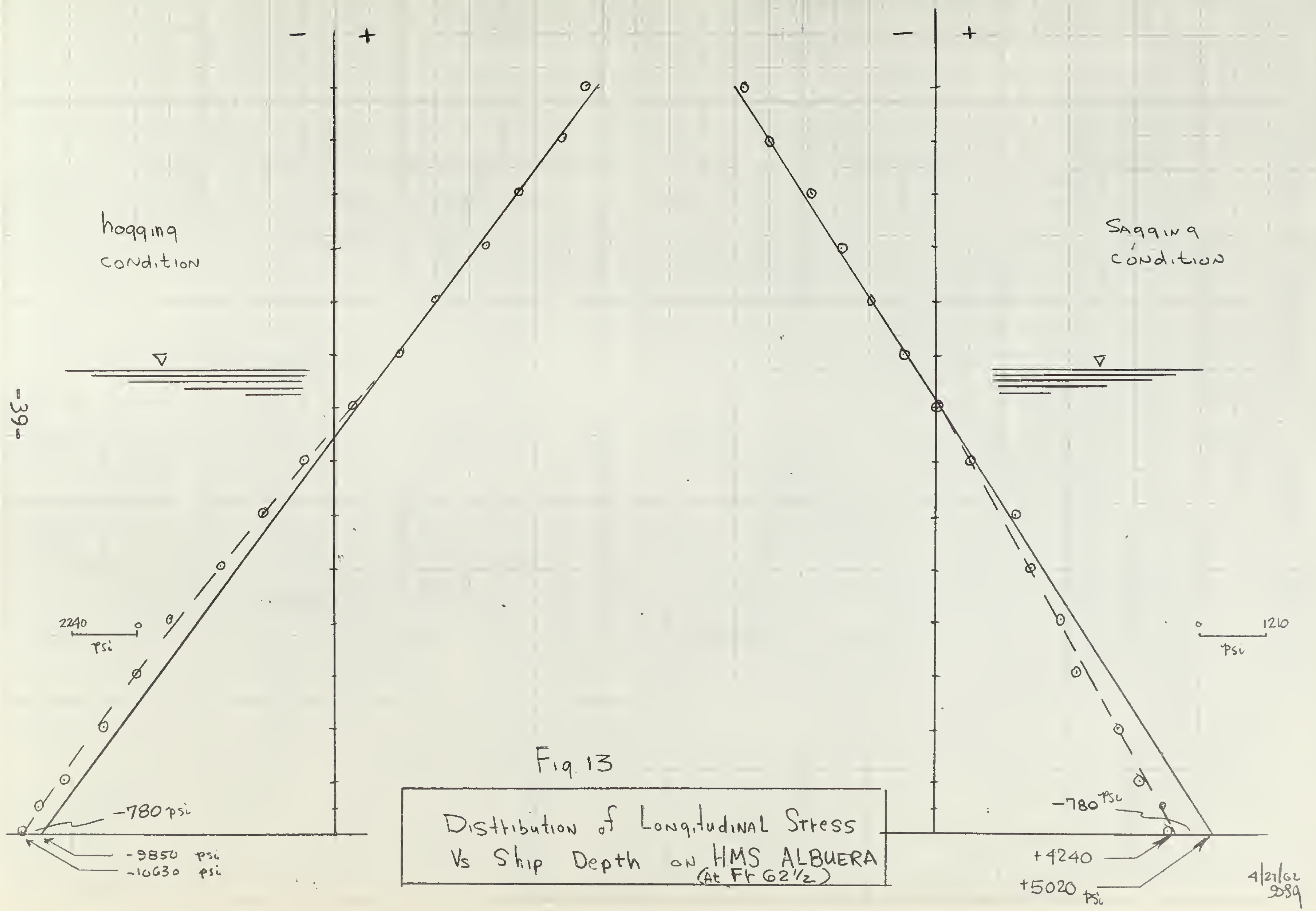



Fig. 13

Distribution of Longitudinal Stress Vs Ship Depth on HMS ALBUERA (At Fr 62½)

Figure 14 Extensional Rigidity Comparison

for DE 1033

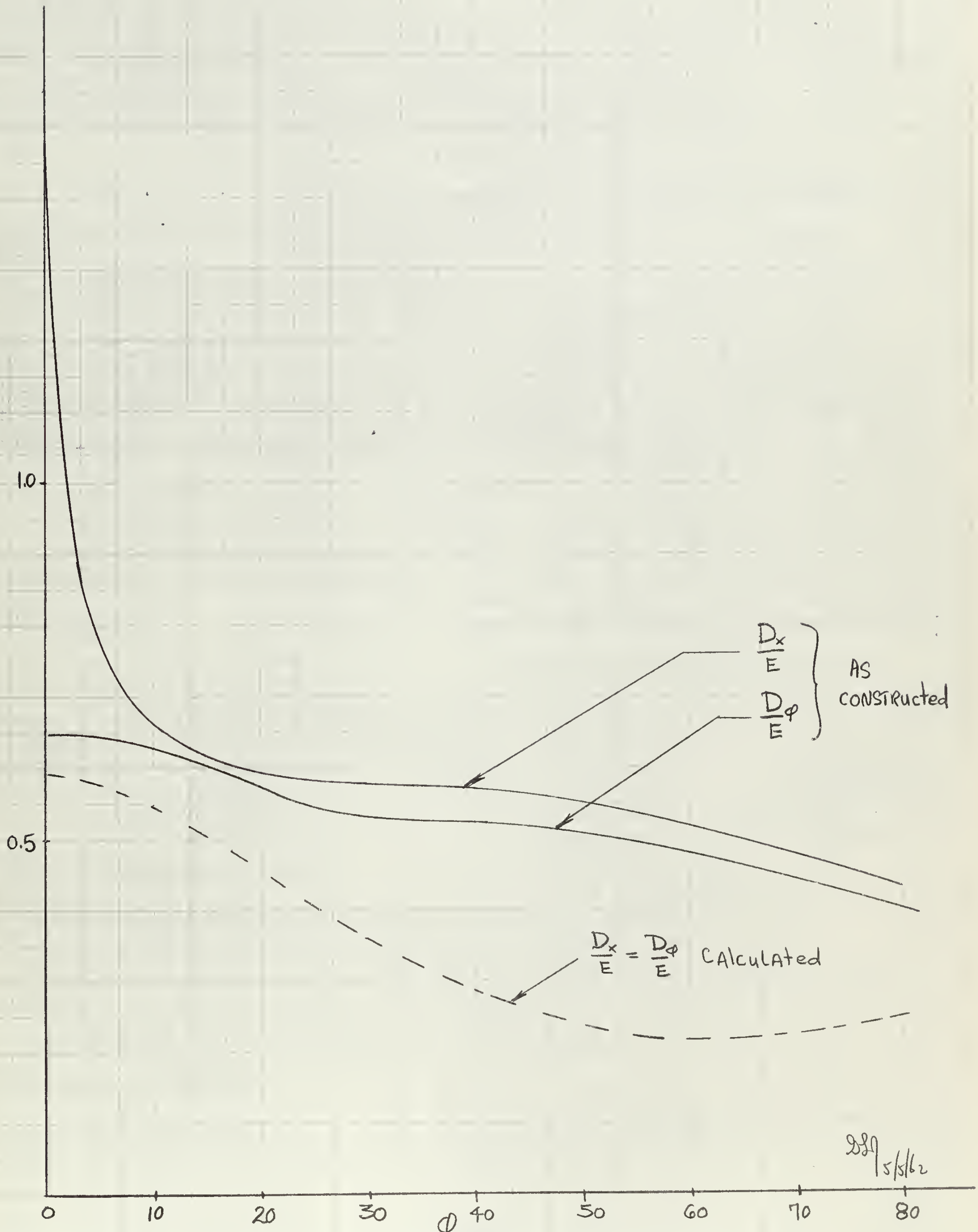
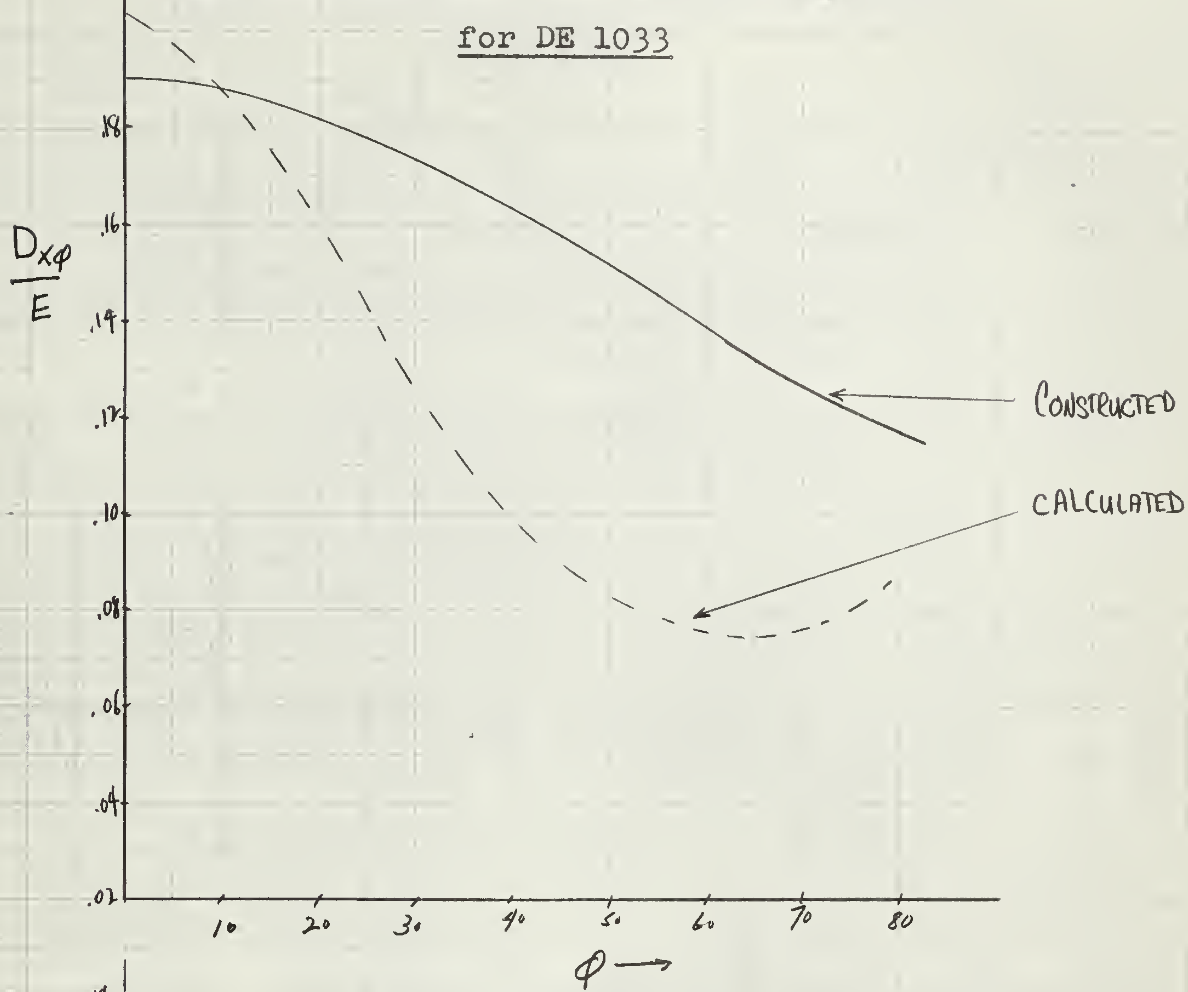


Figure 15 Extensional Rigidity Comparison

for DE 1033



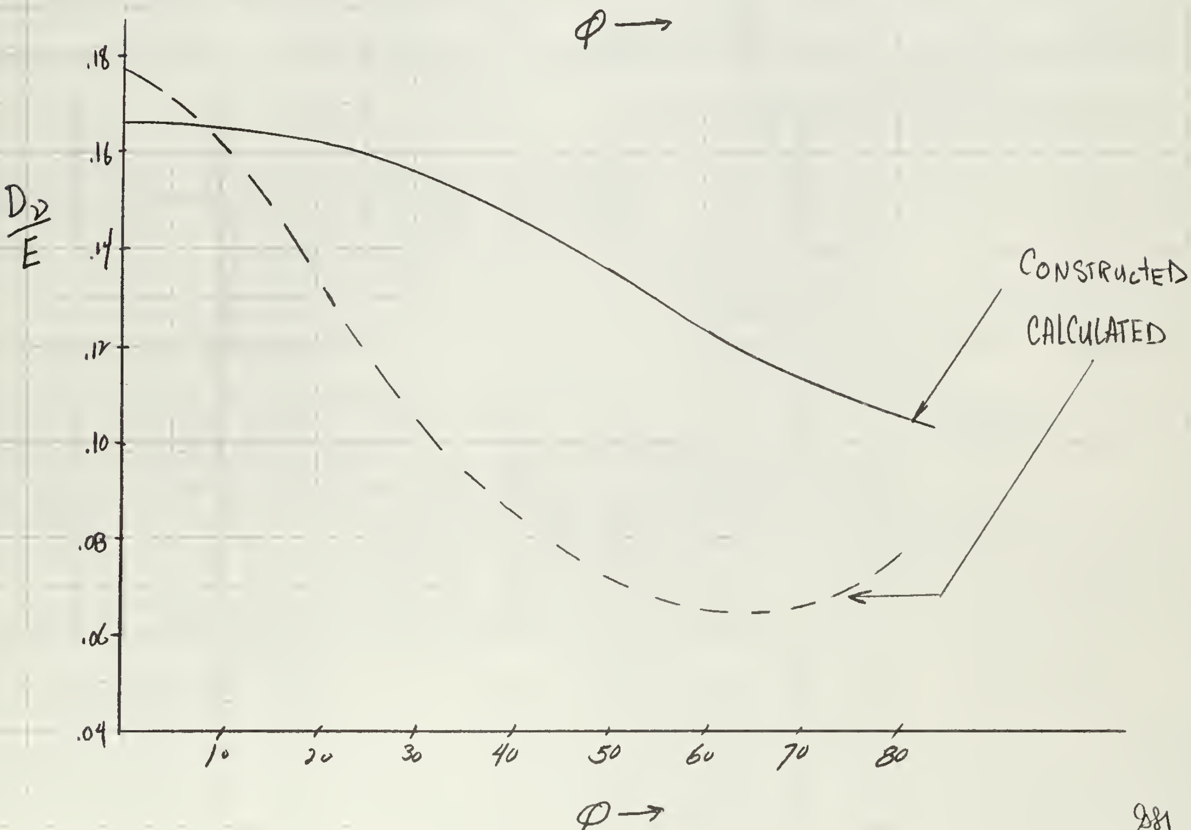




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Figure 16  Extensional Rigidity Comparison

for DD 931

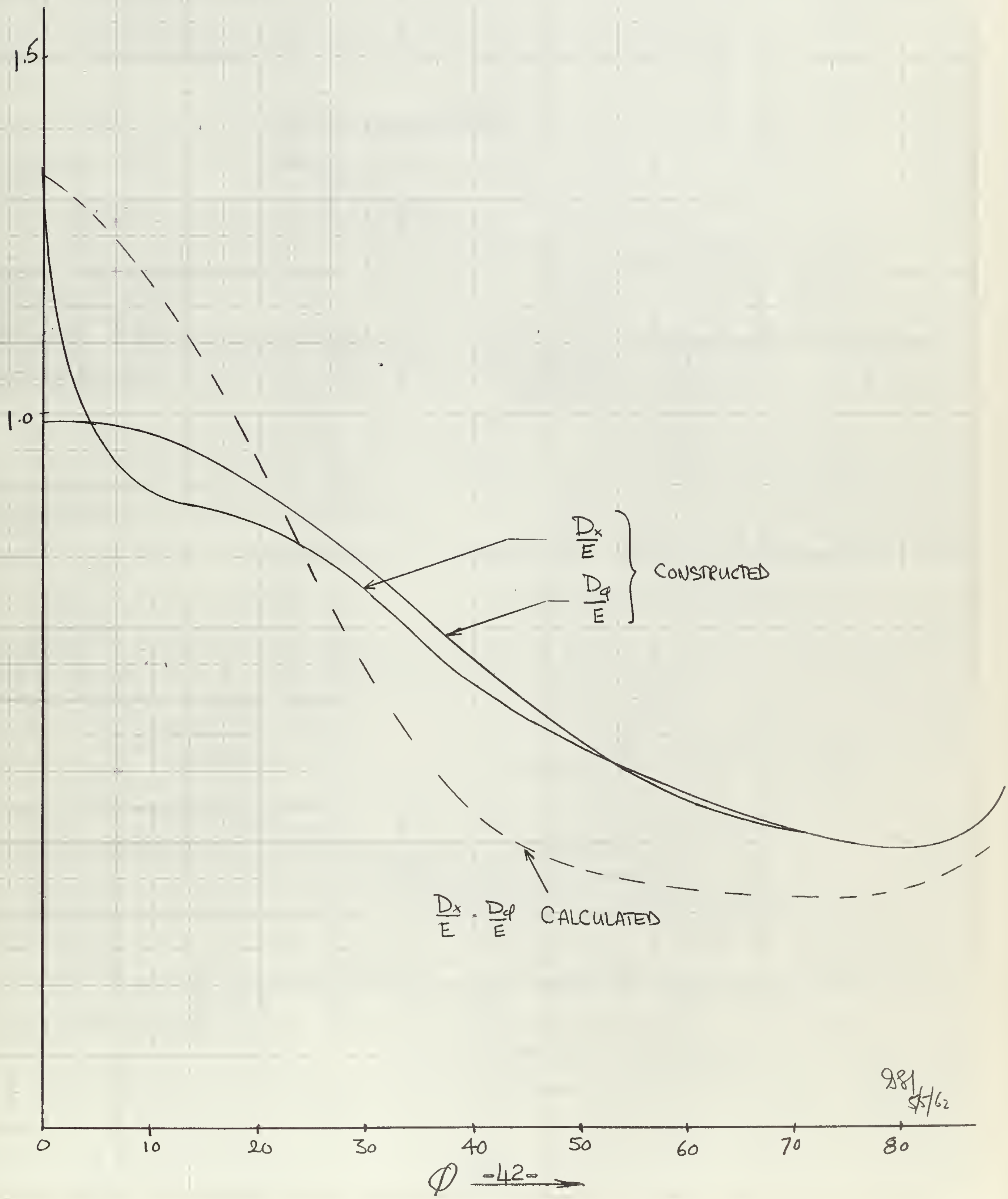
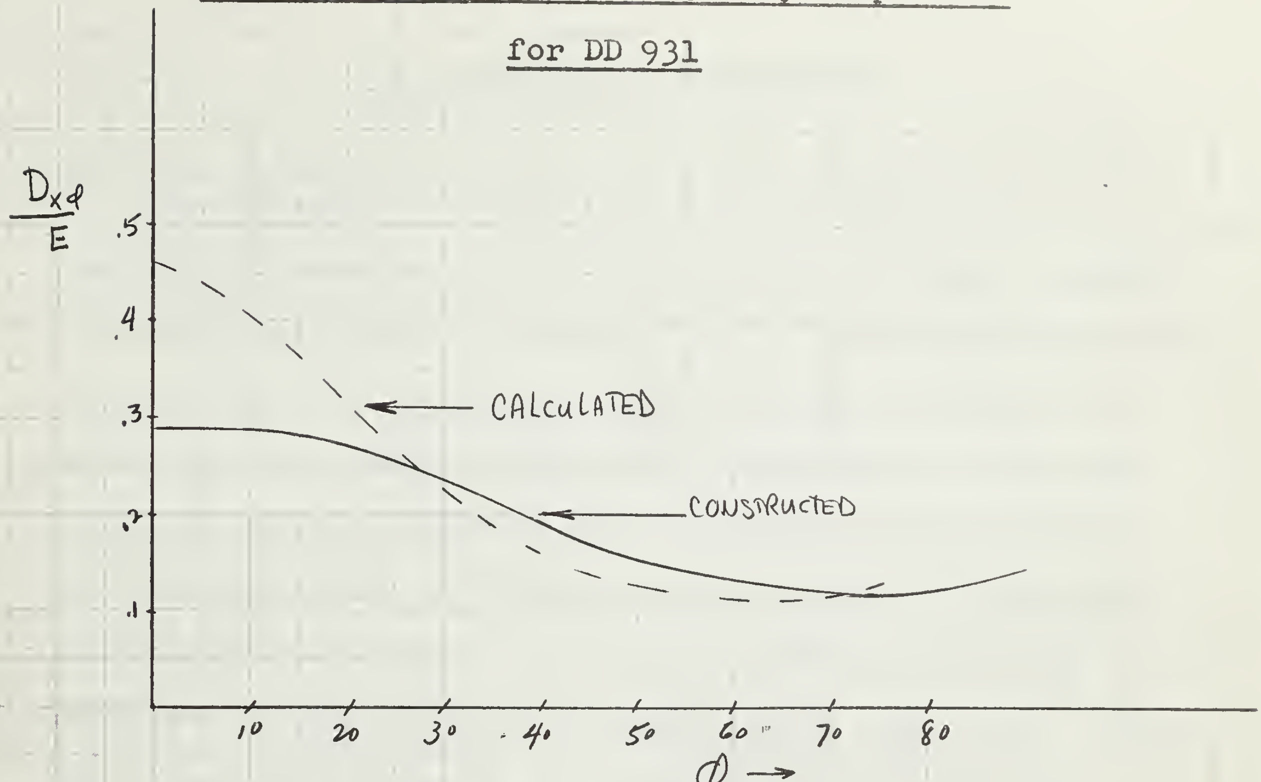
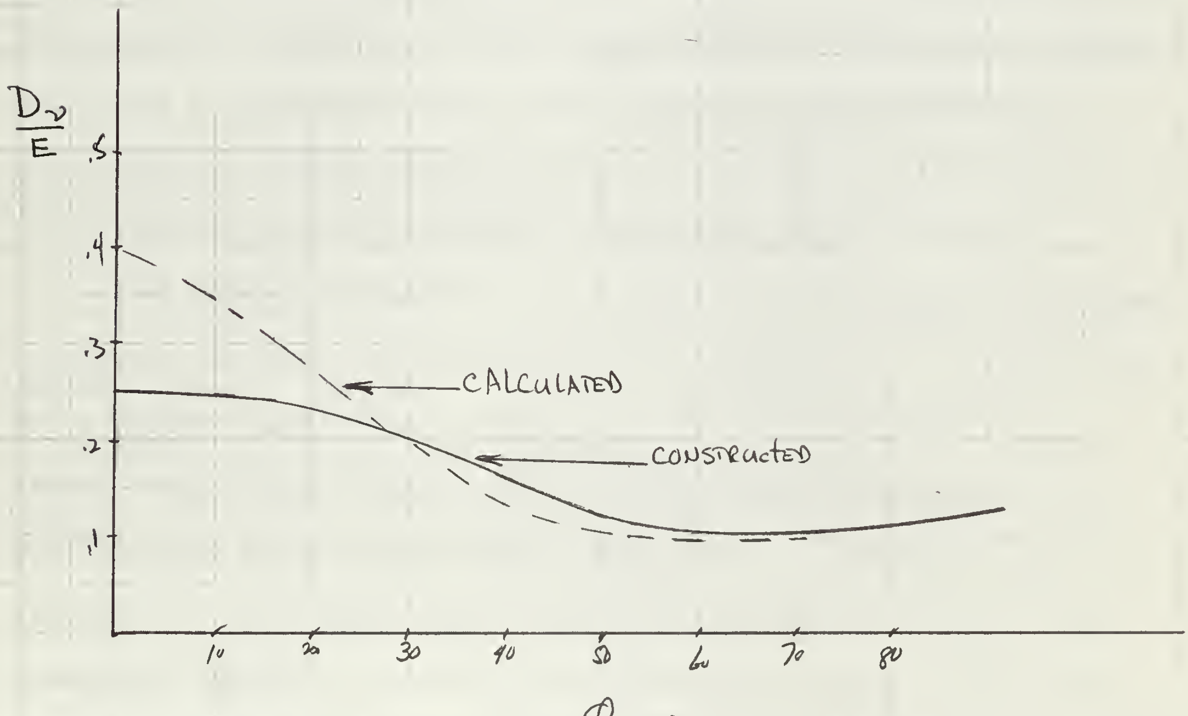


Figure 17 Extensional Rigidity Comparison

for DD 931

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# IV. DISCUSSION OF RESULTS

The inherent complexity of the developed set of partial differential equations precludes a simple solution. An analytical solution appears to be a formidable challange. Perhaps the most promising course of action would be to apply the technique of difference equations and with the aid of a digital computer obtain an approximate solution. The complex nature of the equations should not discourage efforts to solve them, for in their solution lies the complete behavior of the stiffened shell. This last statement needs qualification in that the behavior does not include the effects of concentrated loads, or localized stresses. However, as noted and discussed in Appendix B the general behavior of the shell should be obtained first and then a follow up check made for localized stresses. The shell analysis should not proceed solely on a series of localized stress analyses throughout the structure.

The value of the above equations is naturally dependent upon the validity of the included assumptions. Perhaps the most questionable one of these is the assumption of constant orthotropic rigidity constants with variation in $\phi$. This simply states that the structure is uniformly constructed of the same size structural elements around the section. This, of course, would never be done since the loading varies with $\phi$. Figure 26 illustrates how the actual coefficients do vary with $\phi$. It can be seen that

there are regions of the shell where many are constant with $\phi$. For those that vary slightly, a mean value might be taken. The sharply varying ones are associated with the stiffners in the $\phi$ direction. Perhaps a reanalysis of the transverse strength requirements will reduce these to more smoothly varying values with $\phi$.

Once the equations are solved, the solutions will be in the form of expressions for the deformation as a function of X, $\phi$, and the rigidity coefficients at any given point (with, of course, the effect of the injected boundary conditions being incorporated in the functional expression). Once these deformations are known, expressions can be obtained for the strains, moments, shears and stresses in the equivalent orthotropic shell. The rigidity coefficients may then be determined at any point by assignment of design stresses. These design stresses would, of course, be influenced by considerations of hull girder bending stress and stability of the structure. After all the rigidity coefficients are determined (i.e., the structural member sizes and spacings) they could be fed back to the original expression for deformation along with their variation in $\phi$. A recycling of this process until the coefficients do not change appreciably will give the final structural design. A follow up check for localized stresses and concentrated loadings will complete the process.

The application of this theory and procedure to a photo-

elastic scale model and later a larger steel structural model would prove invaluable. A program of testing, modifying theory and reorienting basic approaches would most probably result in new and improved concepts in naval structural design.

A few comments are in order concerning the hull curvature. The reasons for selecting the analtical curvature (or radius of curvature) used in this thesis are amply covered in Appendix A. However, for additional research in this area, NACA TM 1302, pp. 8 - 16 is strongly recommended. The topic of variable curvature is discussed and fourier polynomials are developed for the radius of curvature expression. Very simple expressions result from the analysis and they could be used to approximate, very closely, any given hull design. It is quite interesting to note that within the scope of preliminary hull design, there exists a design variable unused by the structural analyst. This design parameter is the miship section coefficient (the ratio of the immersed area of the midship section to the product of beam and draft, $\frac{Ax}{BH}$). This coefficient defines the relative fullness at the midship section to a circumscribing rectangle. In effect it is a shape factor which determine the shell section profile. If the stress analyst had some degree of control over this coefficient, a shape or section profile could be determined which would be the most beneficial for the stress

in the hull. The stress magnitude and distribution in any shell structure is a function of its shape or curvature. The encouraging fact is that indeed the analyst can play some part in determining this coefficient to the advantage of the structural requirements without affecting the resistance of the ship. D. W. Taylor (8) demonstrated this fact in 1908 when he conducted model experiments to determine the influence of midship section shape upon the resistance of the ship. His tests were definitive enough for him to make the following conclusion; "for vessels of usual types and of speeds in knots no greater than twice the square root of length in feet, the naval architect may vary widely the midship section fullness with no material beneficial or prejudicial effect upon speed." It must be remembered, of course, that the above statement applies to the shape and coefficient of the midship section for a given area and not to the area itself. Taylor varied the midship section coefficient from 0.7 to 1.1 using 10 models and holding the midship section area constant on all tests. Thus it appears that the structural analyst can accept a given design and have considerable freedom to determine his optimun shell shape (for a given midship area) within the limits of not adversely affecting the prismatic coefficient or general seaworthiness.

The use of membrane theory for the evaluation of stress

in the ships bottom section has considerable merit. Although the stiffened ship structure will carry the stresses according to the distribution of its members, membrane theory will predict what the stresses are that must be carried. There are two basic limitations to the theory. (1) The normal shears, bending moments and twisting moments are neglected, (2) the membrane theory becomes invalid at and adjacent to the boundaries. These limitations are not severe, however, since the quantities of normal shear and bending and twisting moments are normally of a very minor magnitude. (It will be shown shortly how this can be determined quantitatively). The boundaries can be subjected to a separate discontinuity analysis and do not jeopardize the evalution of conditions over the major portion of the shell. To prove that the membrane theory gives is a good representation of the situation, one can proceed in the following manner (1) find the displacements associated with the direct stresses predicted by membrane theory; (2) use these displacements in the general equations which incorporate bending (equations 10); (3) compute the direct stresses from these bending theory equations and compare them to the membrane direct stresses to evaluate the accuracy of the membrane theory.

Figures11, 12, represent the membrane stress distribution in the longitudinal ($\phi$= const) and girthwise (X= const) directions respectively. It must be kept in mind that

these membrane stresses are invalid at or adjacent to any of the shell boundaries (bulkheads and longitudinal junctions with vertical ships side). A matching of reaction forces at these locations is necessary as well as the determination of secondary bending to satisfy compatibility.

The most interesting feature of Figure 11 is the presence of longitudinal compression (- $N_x$ stress) in the shell midway between the bulkheads. If the ship actually had a flat bottom the stress would obviously be a tension. As the bottom curvature increases and becomes a shell, the longitudinal stress does indeed become compression. The direct stress whose magnitude and direction is the most sensitive to shell shape or curvature is the longitudinal direct stress, $N_x$. This is not a conclusion of this thesis but rather a standard fact of shell structure behavior (a fact which is largely responsible for the widespread success and popularity of concrete shell structures). This is of interest to the naval structural engineer since it is this particular stress which is combined with the hull girder bending stress. The recognition of this stress should render ship structural stability considerations more realistic.

Figure 13 represents a plot of the longitudinal stresses as measured on the destroyer HMS ALBUERA (9) in the hogging and sagging conditions. The test section (fr. 62½) was

located in the midship region and about halfway between transverse bulkheads. Considering the observed stresses in the hull above the waterline, a linear relationship is readily established. When the underwater portion of the hull stresses are examined it can be seen that they exhibit a variance from the established linearity (the linearity established the upper portion of the hull must be the same as for the lower since neither moment nor section inertia has changed in going to the lower portion). This departure is obviously due to the effect of the hydrostatic loading on the underwater shell or hull. It will be noted that this effect augments the negative hogging stress and reduces the positive sagging stress. In other words the hydrostatic loading on the hull produces a longitudinal compression ($-N_x$). This agrees with the theoretical plot of $N_x$ in Figure 11. The longitudinal compression effect under hydrostatic loading would be no surprise to the civil engineer who works with shells. It would, however, be unexpected by a naval structures analyst who limits himself to the study of small panels and predicts only a flat plate tension effect due to hydrostatic loading. While it is quite true that each panel as such is subjected to a tension producing load, this is only a minor superposition on the more general shell behavior which is a compression. These localized stresses can be checked in the later stages of design.

Figure 12 illustrates the distribution of direct stresses around the girth ($X = $ const) for two selected destroyer types.

The section portrayed is at the left transverse bulkhead of the basic shell ($X = -L/2$). This particular section was selected for purposes of obtaining a design shell thickness (and its variation with $\phi$) since its thickness requirement is greater than any other intermediate section for any given $\phi$ value. This was an unfortunate choice, however, as a forthcoming discussion will indicate. The direct stresses of Figure 12 were used in calculating the equivalent shell thickness and hence extensional rigidity constants (tables 4 , 6, ). These coefficients are compared with those actually existing in the ships as a result of their construction. As was previously stated, the membrane theory will give reasonably good results everywhere except the boundaries. This unfortunately was the necessary region (trans. BHD) to analyze since it produced the controlling equivalent shell thickness. The evaluation of the direct stresses could have been and should have been improved by using the procedure indicated on page 48. Since the extensional rigidity coefficients were calculated from $t_e$ which in turn was calculated from the direct stresses; additional corrective action on the direct stresses produced would result in a different and better correlated set of extensional coefficients. It is surprising that at this worst possible location for membrane theory, the degree of agreement between constructed and calculated is still reasonably good. A corrective cycle on the obtained membrane stresses would result in a general increase of varying degree for the rigidity coefficients at all values

of $\emptyset$ (since all $\emptyset$ values are on the boundary). The difference in relative variation of calculated from constructed values in the two ships selected is due to several factors which characterize each ship. It is believed, however, that one of the major causes is the fact that shell curvature of DD 931 (considered similar to DD 952) resembles a cycloid to a lesser degree than does DE 1033. This suggests the importance of selecting the most accurate analytical model for hull curvature. Other causes for non similarity of the ships plots are possible opposite departures from the assumed conditions in the membrane equations (e.g. $\sigma_1$ stress estimate, factor of safety used and degree of fixity at ends). Figures 15,17 , merely represent the comparison of the calculated and constructed ship hull thickness since these rigidity coefficients are just a product of a constant and the thickness.

The approach used in this thesis was not expected to produce any radical changes in structural design. It was intended to achieve a more rational design attitude by treating a section between bulkheads as a shell and then combining these results with the overall ship bending stress. An expected result was a slight reduction in structural members as determined by current design. With due regard to the previously discussed obstacles and handicaps involved, the plots of Figure 14,15,16,17, indicate an apparent trend in this direction. The determination of strucutral members can-

not be made on extentional coefficients alone. A recycling of the membrane solution as noted on p  will give magnitude of moments in the equivalent shell. The individual structural members for the ship may then be determined by a combined consideration of the extentional rigidities required, the moments in the shell and the localized panel stress.

Unfortunately neither one of the basic approaches used in this thesis (i.e., shell bending theory by differential equations and membrane theory) has been followed through sufficiently to give definitive quantitative conclusions. The investigation, however, has been quite profitable in that the ships hull has been analyzed in a manner more appropriate to the geometry of the structure. Such an analysis can and, indeed, did reveal information unobtainable by an otherwise less realiztic approach. Specific areas for additional study and analysis have been clearly indicated.

# V. CONCLUSIONS

1. The resulting partial differential equations of the orthotropic shell behavior are complex and even after simplification are not amenable to an analytic solution.

2. A rational structural design procedure can be established by using the partial differential equations solution and the orthotropic rigidity coefficients (see page 45).

3. The analytical expression for the hull radius of curvature must be uncomplicated and yet representative of the actual hull form being analyzed.

4. The midship section coefficient or the midship section shape is a design parameter which can be used within reasonable limits to benefit the structural design with no prejudicial effects on hull resistance or ship speed.

5. Subject to its characteristic limitations, membrane theory can be used in the preliminary analysis of ship structures. Iterative corrective procedures can be used for improvement of solution and/or estimation of existing error.

6. A longitudinal compression results from hydrostatic loading on a ship hull that possesses curvature. The magnitude and distribution of the compression around the

girth of the ship is sensitive to the hull shape or variation of curvature.

7. Useful extensional rigidity coefficients can be determined by membrane theory when iterative corrections are included.

8. Current design methods produce hulls which have a girthwise bending stiffness which is five times as large as the longitudinal bending stiffness.

9. The different and more appropriate approach used in this thesis for the analysis of ship structures with hull curvature has produced:

(a) A specific knowledge of items within this analysis requiring further study and refinement.

(b) A more discerning attitude towards the structure being designed.

(c) Suggested design procedures utilizing the methods and techniques discussed in this study.

## VI. RECOMMENDATIONS

1. Computer solutions to characteristic partial differential equations of the orthotropic shell structure should be devised, programmed and tested for merit in relation to current or other proposed design concepts.

2. Optimization of ship structural members and their arrangement should be on the basis of an orthotropic shell and not on a series of panels or current design approximations.

3. A study should be conducted to determine if a defineable relationship exists between the longitudinal and transverse strength requirements of any given design. The current ratio of five to one on girthwise flexural stiffness to longitudinal stiffness appears excessive.

4. Experimental verification of all developed theory and techniques should proceed concurrently to aid in providing realistic results which are of a practical significance. Experimentation could commence economically by photoelastic stress analyses of small scale models. Tests on successively larger steel scale models using

adequate instrumentation should proceed if early results indicate that further testing can be of value.

5. A short study should be made to determine an analytical expression for the radius of curvature of any given design. The expression must be uncomplicated or capable of being simplified for the mathematical operations which must be made on it. (NACA TM 1302 pp. 8 - 16 is recommended for this study).

6. Since the mathematical analysis of a circular orthotropic shell is quite complete, one possible approach is to break the shell into a finite number of segments represented by constant radii of curvature (circular), make the analysis and join the segments by making them compatible at all junctions.

7. A strongly recommended topic for study is the variation of midship section coefficient or shape for a given midship section area and the determination of the optimum shape for structural design. The only major constraints will be the maintenance of the designed prismatic coefficient and desired seaworthiness characteristics. Within these constraints, however, there is latitude for increasing structural efficiency.

8. A modified membrane theory should be developed for

the preliminary analysis of ship structures. Since the solution improves with iteration, it is quite amenable to machine computation.

9. A method should be devised for readily determining the magnitude of longitudinal compressive stress which is produced by hydrostatic loading on all hulls with curvature. This value should then be included in any deliberations on structural stability.

10. A complete analysis of the stability of the orthotropic shell between bulkheads should be combined with this thesis to provide the complete ingredients for a new concept of ship structural design.

11. A relieable solution to the partial differential equation of the orthotropic shell will require a knowledge of the shell boundary conditions. It is apparent that a study devoted soley to this subject would be of considerable value.

# VII. APPENDIX

## A. HULL CURVATURE

The curvature of the hull in the girthwise or circumferential direction is one of the most significant factors in this investigation. The author feels that the magnitude and variation of hull curvature for ships of the destroyer type has a significant effect upon the structural behavior and therefore should be given full consideration in the structural analysis of these hulls. The section of the ship considered was in the midship section region. There will normally be no longitudinal or fore and aft hull curvature in this middle portion of the ship.

The argument could be made that the hull girthwise curvature is really negligible if a basic panel (bounded by frames and longitudinal stringers) is selected for analysis. The weakness of this argument, however, is that the hull's general response is not on a panel basis but on a much larger and continuous region (i.e., the portion of the hull structure between transverse bulkheads) wherein any flat plate or flat bottom assumption would be unrealistic. Additional comments on the relative size of hull portion analyzed will be reserved until Appendix B.

The task of introducing the hull curvature of a specific type of ship, or of a given design within a type, is a difficult one. The analytical expression of curvature selected must be representative of the design being studied. It must also be a reasonably uncomplicated function capable of being

subjected to the necessary mathematical operations without excessively expanding the complexity of the resulting analysis. The representation of ships hulls by polynomial expressions is not appropriate since the desired expression is one giving the radius of curvature as a function of the girth angle ($\phi$).

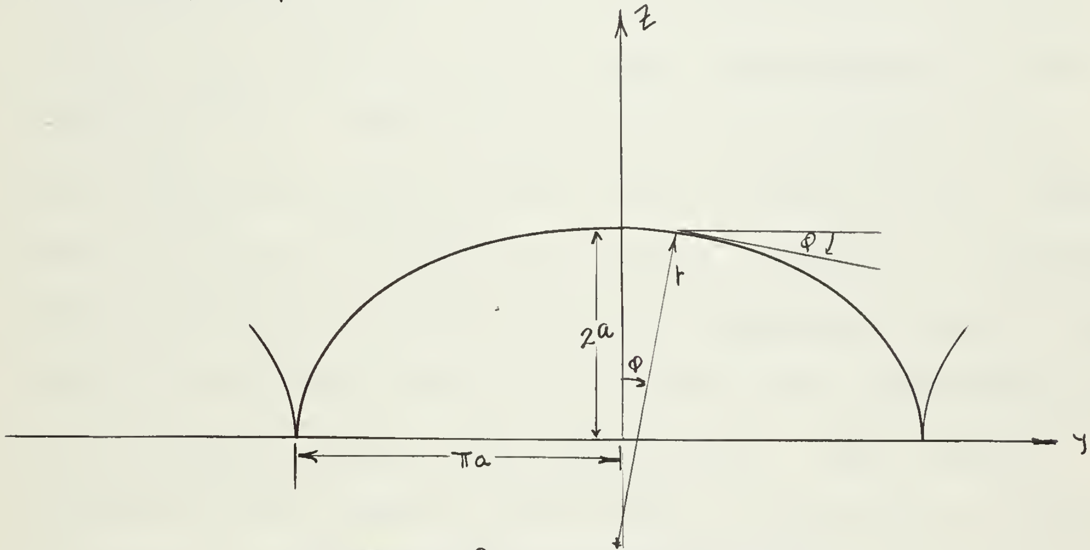


Figure 18 Cycloid Profile

The analytical expression finally selected as satisfying the above requirements was the equation of the cycloid $r = 4a \cos \phi$ (Figure ) where the depth of the curve (draft) is 2a and the span (beam) is $2\pi a$. ("a" is simply the radius of the generating circle). This type of curve or section profile results in a span to depth ratio (or B/H ratio) of $\pi$ . The B/H ratios of the ship type considered is indicated in Table 2.

| | DE 1033 | DD 952 | DLG 9 | DLG(N) | CG(N) | CA 134 |
|---|---|---|---|---|---|---|
| B/H | 3.16 | 3.13 | 3.14 | 3.02 | 3.25 | 3.14 |

TABLE 2. B/H Ratios of Ship Types Considered

The $\pi$ B/H ratio of the cycloid does not in itself qualify this curve as being a good representation of the destroyer hull. The curvature and its variation with $\phi$ must correspond sufficiently close to the ship hull shape. This is the most significant factor. It must be remembered that coincidence of the analytical expression and the ships hull is not of importance (i.e., the r, $\phi$ relationship should not be misconstrued to be a polar plot). The relative equality of the values of the radii of curvature at given $\phi$ values is the significant measure of the expression's adequacy.

The parametric equations of the cycloid in $\phi$ are,

$$y = a(2\phi + \sin 2\phi) \tag{22a}$$

$$z = a(1 + \overset{\cos 2\phi)}{\cancel{\cos\phi)}} \tag{22b}$$

The cycloid section can then be fitted to any given design by letting $a = H/2$ and plotting

$$y = a\left[\cos^{-1}\left(\frac{z-a}{a}\right) + \sin\cos^{-1}\left(\frac{z-a}{a}\right)\right]$$ obtained from

an elimination of $\phi$ in equations 22. These plots have been made in figures 20, 21 , 22 , 23 , along with the plots of the curve produced by fitting an elliptical section to each actual

hull shape selected for comparison.

A study of the comparative plots of hull shape and analytical expressions reveals that the elliptical section possesses a greater resemblance to the curvature (or radius of curvature) of the actual hull than does the cycloid (note how the DE 1033 curvature is virtually coincident with the elliptical section). However, the analytical expression for the radius of curvature of the ellipse is

$r = \dfrac{a^2b^2}{a^2\sin^2\phi + b^2\cos^2\phi}$ where a and b represent the major and minor axis respectively. This expression for the radius would not be suitable for the mathematical operations involved in the shell theory of the hull. It is also of interest to note how the hull form of the CA 134 is not appropriate for this analysis since it deviates excessively from curvature given by either expression. CG(N) hull form would also be considered unsuited for the cycloid approximation.

The obvious disadvantage of the cycloid section is the unfortunate fact that the radius of curvature goes to zero at $\phi = \pi/2$. This is inherent in the nature of the cycloid curve since at the extreme points of the span the generating curve changes the sign on the radius of curvature and, at this point, it passes through a zero point. This is not a major disadvantage, however, since it occurs at the region where the curved shell joins the flat plate portion of the

hull (the boundary of the shell section). This particular region requires a separate analysis regardless of shell theory employed (bending or membrane theory). In the former case, rather undefineable boundary conditions exist along with compatibility obstacles and in the latter case membrane theory never does account for conditions at the boundaries. Therefore, it is of no consequence that the radius of curvature expression given by the cycloid is invalid at the boundary of the shell since the shell theories are also in question at that region. A pertinent subject for investigation would be the study of the discontinuity stresses present at this junction. Results could be correlated with this thesis which purports to treat only the bottom section of the ship not at or adjacent to the boundaries.

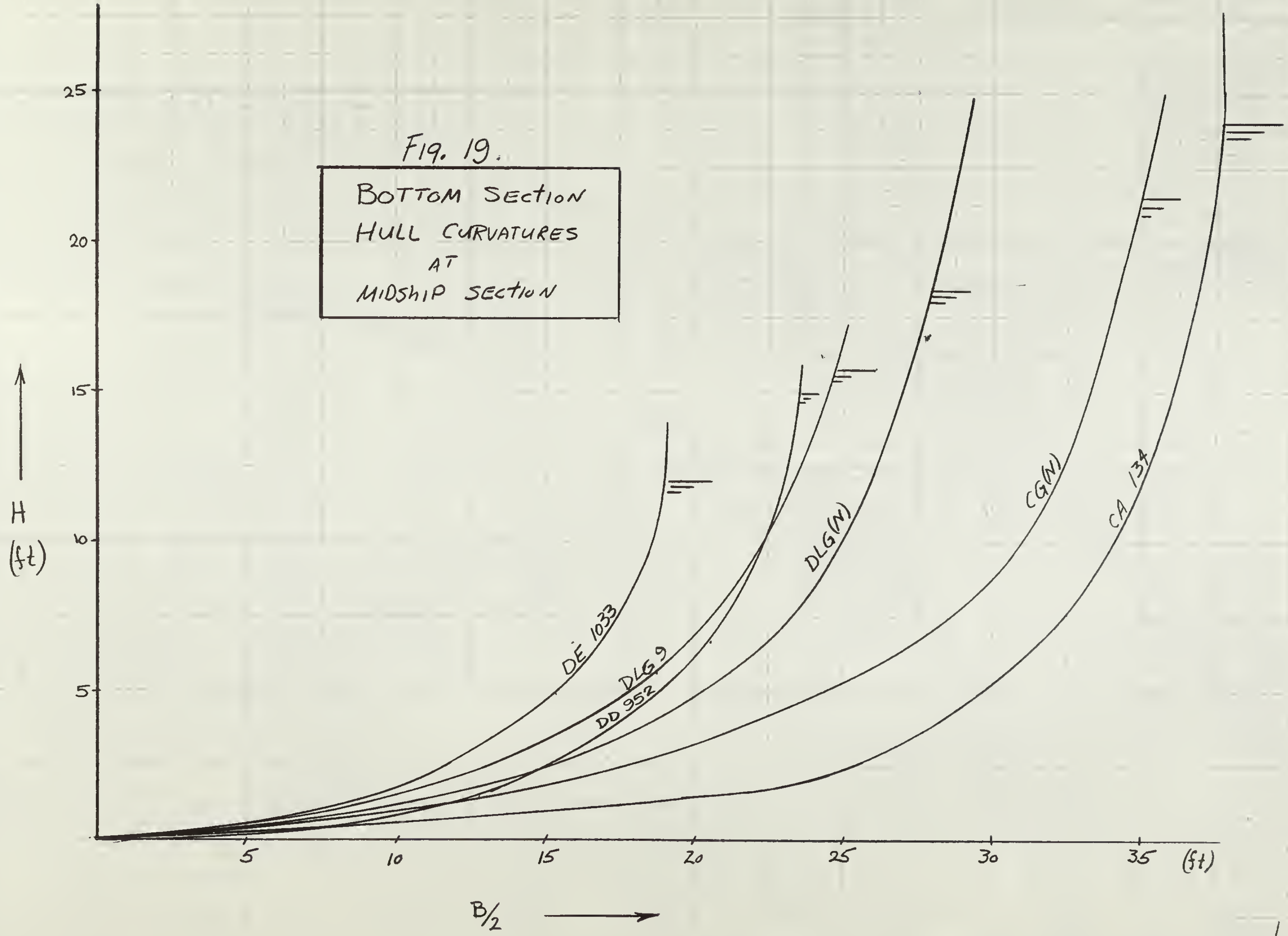


Fig. 19.

BOTTOM SECTION HULL CURVATURES AT MIDSHIP SECTION

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Fig. 20 Comparison of Hull Curvature with Analytic Expressions

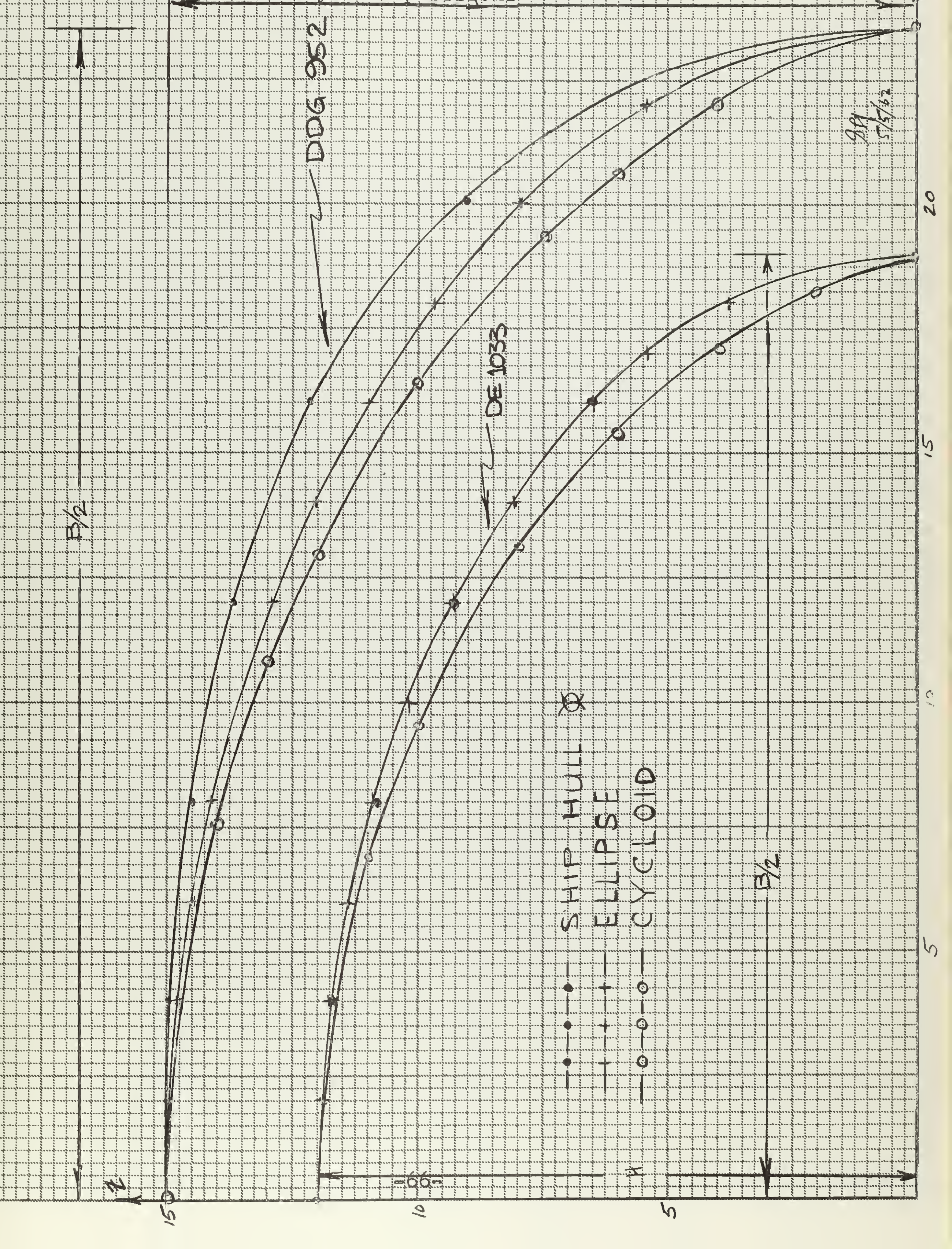

**Fig. 21  Comparison of Hull Curvature with Analytic Expressions**

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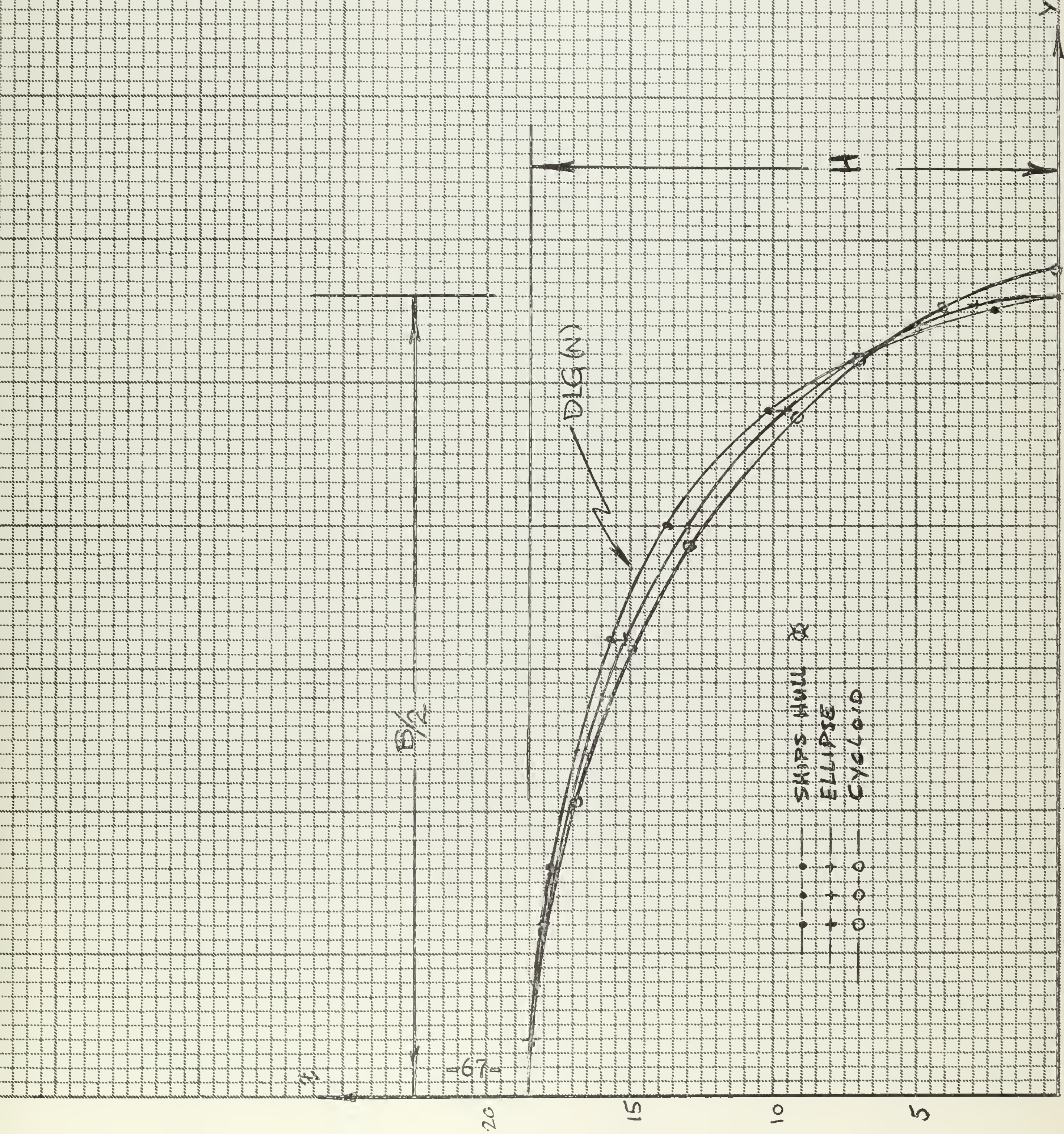

Fig. 22 Comparison of Hull Curvature with

Analytic Expressions

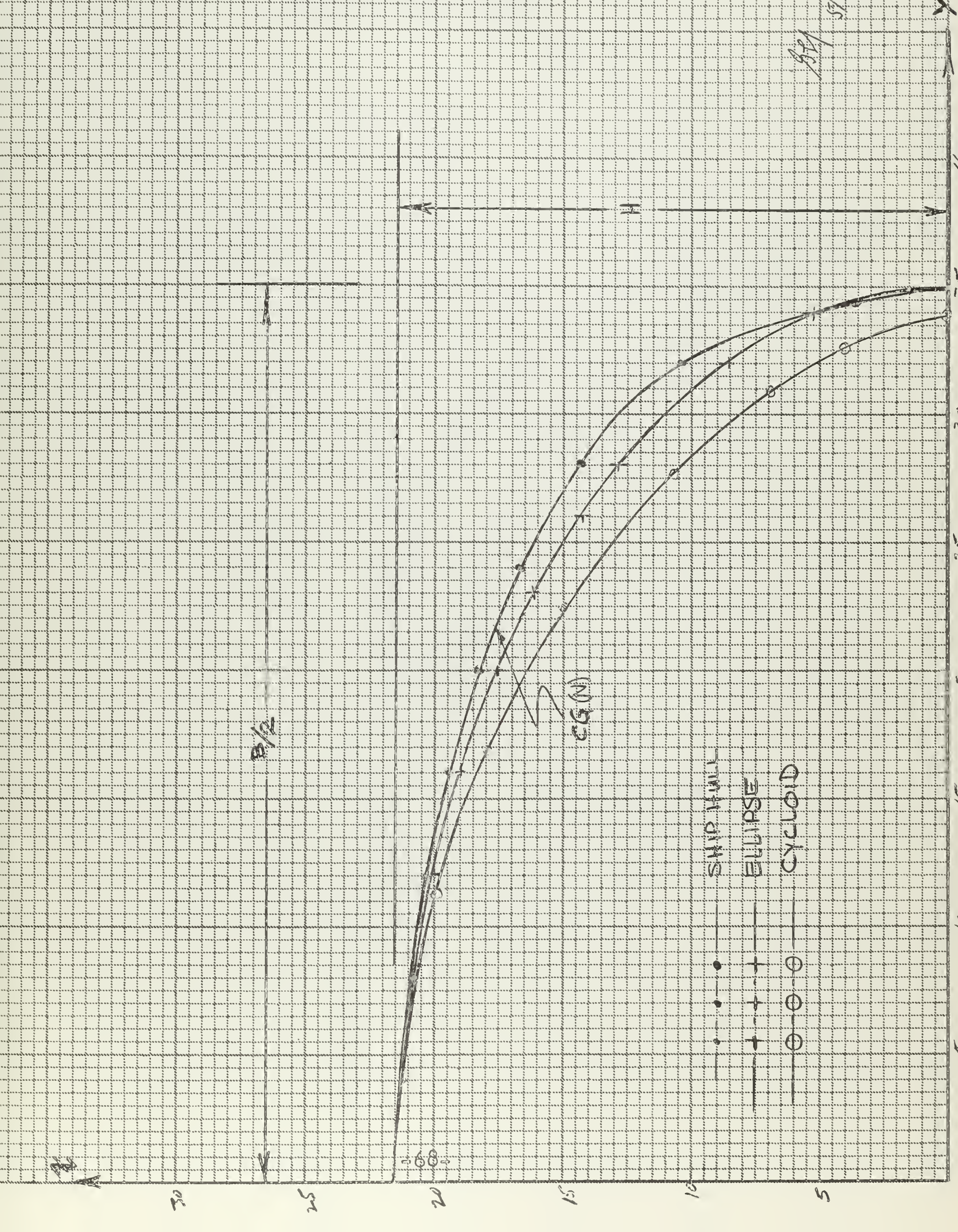

Fig. 23 Comparison of Hull Curvature with

Analytic Expressions

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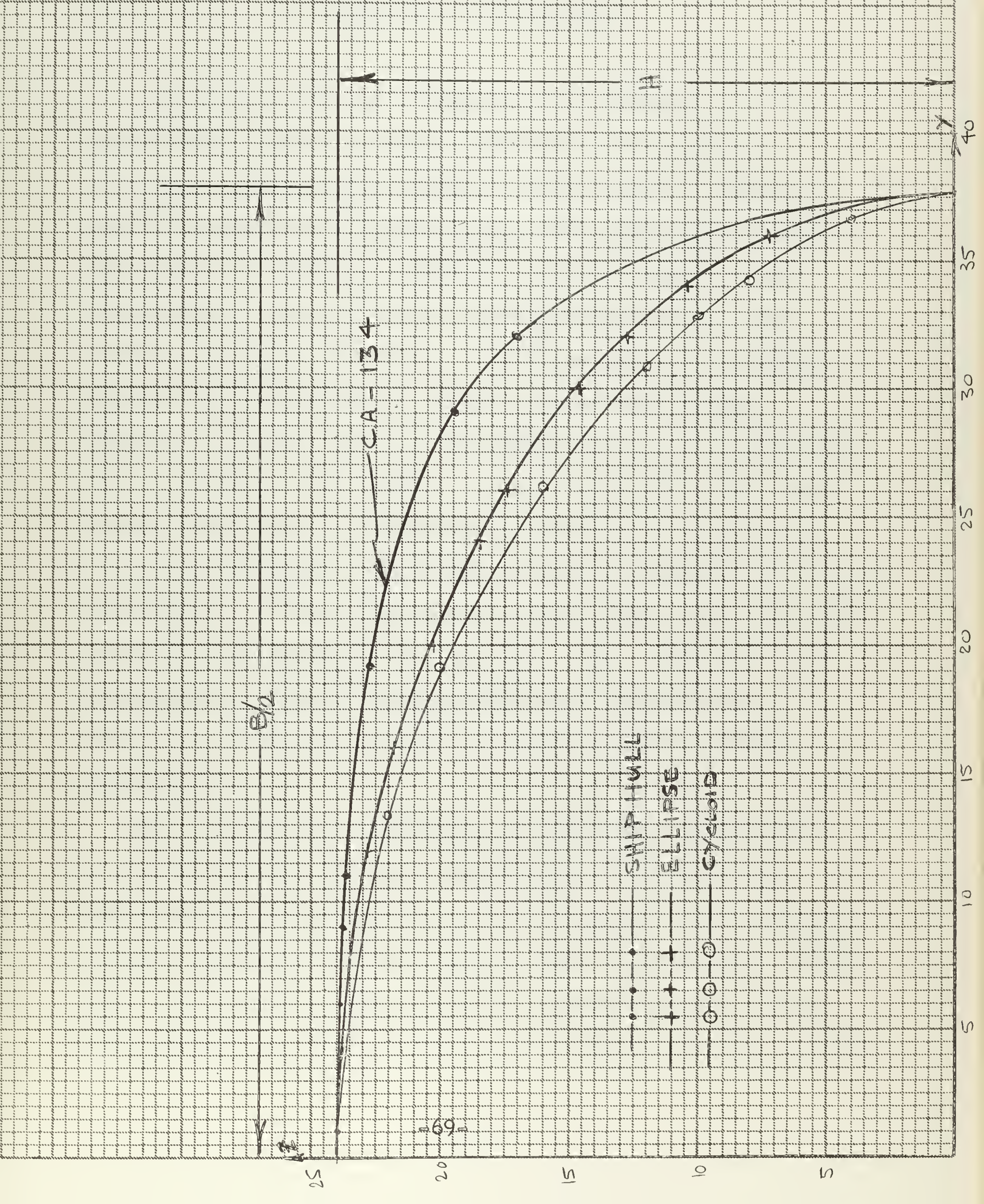


## B. ORTHOTROPY OF SHIPS HULL

As a ships hull is fabricated, the elements of construction (shell plating, longitudinal stringers, transverse frames) are arranged in such a manner that the resulting structural elastic properties (extensional, bending, shear and twisting rigidities) are not equal in all directions as they are for an isotropic structure. The varying of elastic properties with direction, or the anisotropy of the structure, is somewhat simplified by the symmetry of construction in the longitudinal and circumferential directions. This special case of anisotropy where the elastic properties are different in mutually perpendicular directions is termed orthotropy and such structures are called orthotropic. It should be noted at this point that the basic material of construction is isotropic. The disposition of this material creates the anisotropy and therefore the structure can quite correctly be called constructionally orthotropic.

The reaction of a structure to any given loading will be a function of its elastic properties. It would seem then that a realistic analysis could be made if the structures elastic properties and their change with direction are recognized. This can be accomplished by considering an "equivalent" structure without stringers and frames but whose elastic

properties and general behavior under load resembles sufficiently the original structure.

The advantage of this approach is that the continuity of the structures behavior is retained and the resulting analysis is superior to one in which conservative attempts are made to account for the complex interactions existing between structural elements such as frames, longitudinals and shell plating. A critical question arises at this point - what is the characteristic mode of deformation of the stiffened hull? Does it deform under load to form a smooth half wave length between bulkheads as in Figure 24a or is all the deformation confined to the shell surface between stiffening members as in Figure 24b ?

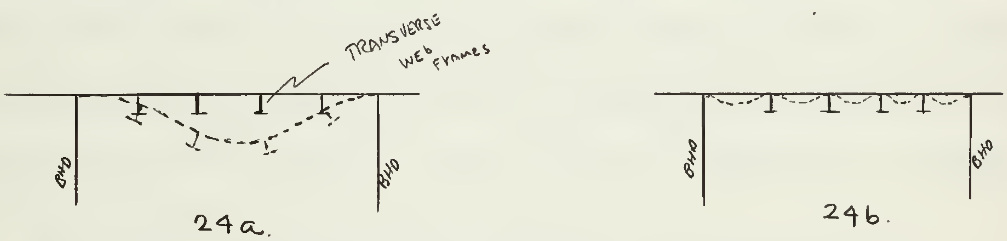



Fig. 24 Stiffened Hull Modes of Deformation

The answer is clearly a function of the number and size of the transverse frames. Lundgren (7) is satisfied to accept the former version of deformation (Figure 24a), whenever the number of transverse frames between bulkheads equals or exceeds four. There are panel advocates, however, who insist on the latter mode (Figure 24b) as being the predominant one which accounts for most of the strain energy of deformation regardless of the number of frames.

For the destroyer type having approximately 4 frames

between bulkheads (in the middle half of the ship), the author feels that the normal mode of deformation is probably a combination of smooth half wave length deformation with smaller panel deformation superposed on it as in Figure 25 .

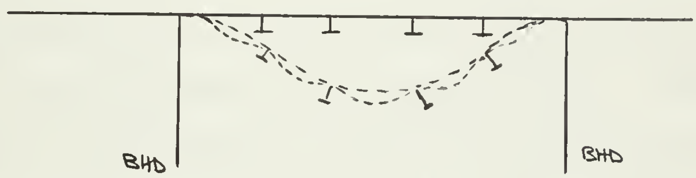



Fig. 25 Combined Modes of Stiffened Hull Deformation

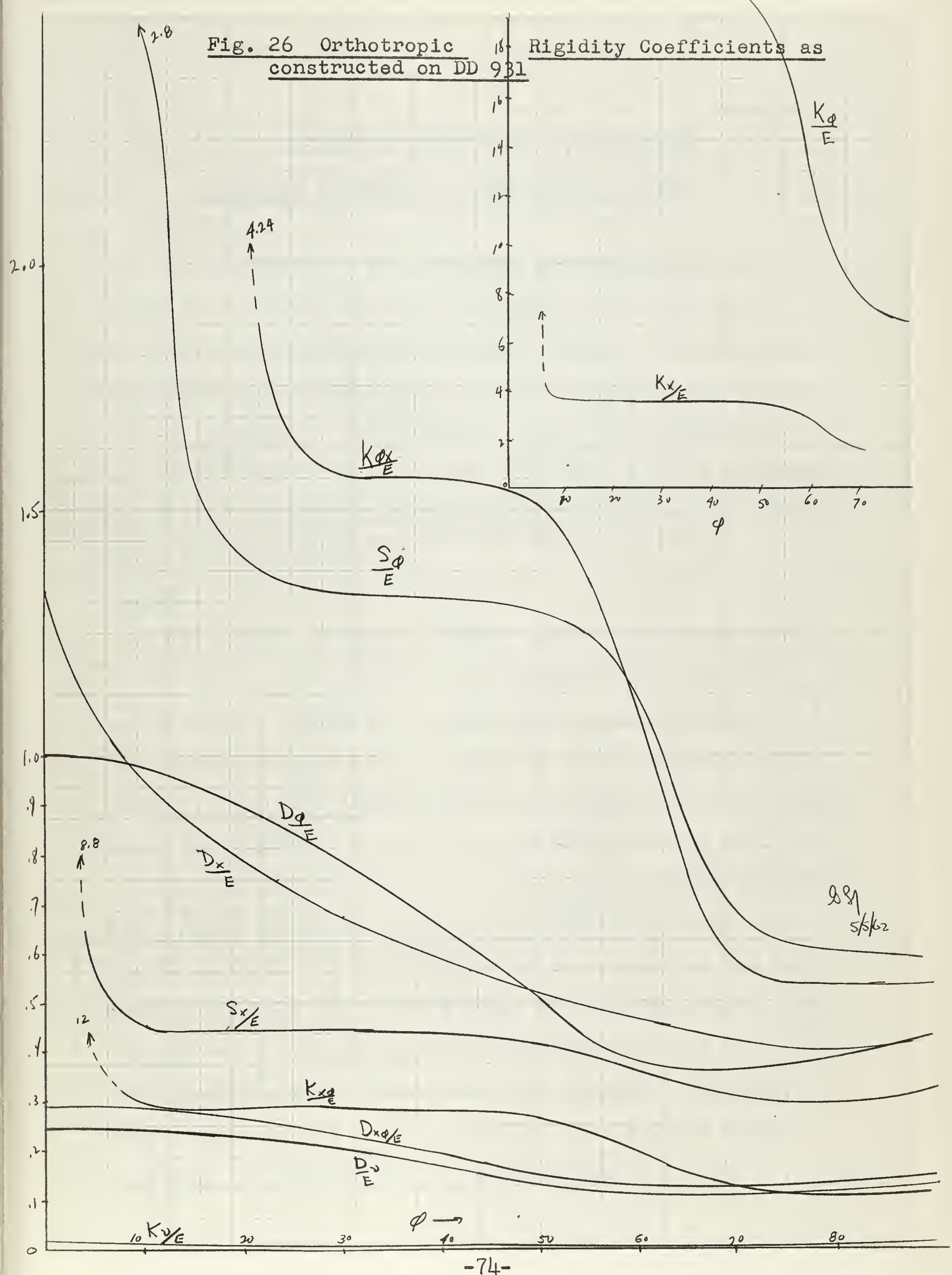
Perhaps many of those who can accept only panel deformation are misled by the physical appearance of ships hulls which have the so called "hungry horse" look or a permanent deformation on all panels making the shell appear as though it were shrunk fit onto the framing system. These effects could certainly have originated from shrinking of the shell around faying surfaces welded to the hull or possible a dynamic loading effect of the ocean waves.

The deformation of Figure 25 (which is the authors version of the situation) illustrates that continuity of the structure between bulkheads must be considered in analysis. The extent of this significant structural section also requires a full consideration of shell curvature. One handicap of the gross section approach which is also obvious from Figure is that the localized stresses created by the superposed panel deformation would not be immediately determined. A final check of the solution should incorporate a method of

accounting for these localized stresses.

The orthotropic rigidity coefficients (equation 14) have been obtained and defined by the superposition of the elastic laws of the isotropic shell and the mating gridwork frame. A better appreciation for these may be achieved by an examination of these coefficients on an actual hull. Figure 26 illustrates the magnitude and variation of the coefficients for the DD 931 class destroyer. The data for these plots is in Appendix D .

Fig. 26 Orthotropic Rigidity Coefficients as constructed on DD 931



## C. BOUNDARY CONDITIONS FOR USE WITH PARTIAL DIFFERENTIAL EQUATIONS OF SHELL

The solution of the developed partial differential equations (15) will eventually require the stipulation of appropriate boundary conditions. These conditions will consist of a rational approximation of (a) the degrees of fixity existing at the bulkheads and at the junction of the curved shell and flat ships side and, (b) the magnitude and distribution of the loading which exists at the aforementioned boundaries.

Fixity:

The degree of fixity which exists at the boundaries of the shell could indeed be the subject for a separate and absorbing analysis. Vedeler (1) suggests a method for accomplishing this. It would be of questionable value however, to pursue this subject in any depth in this thesis since the immediate problem of determing a method of solution to the differential equation is the major obstacle. For a first approximation then, the following conditions are postulated. At the junction of the shell and the bulkhead, the fixity can be considered to be 100% clamped. The presumtion is a natural result when one considers that the structure is a shell continuous over supports (bulkheads) with continuous and equal loading on each side of these supports. An examination of the bulkhead $N_x$ stress

distribution in Figure 11 would also lead one to assume that a full clamping condition prevails. The fixity condition along the longitudinal boundary is a bit more elusive. Once more, as a first approximation, it will be assumed as being simply supported and capable of movement in the athwartship direction. The condition can be represented by Figure 27 .

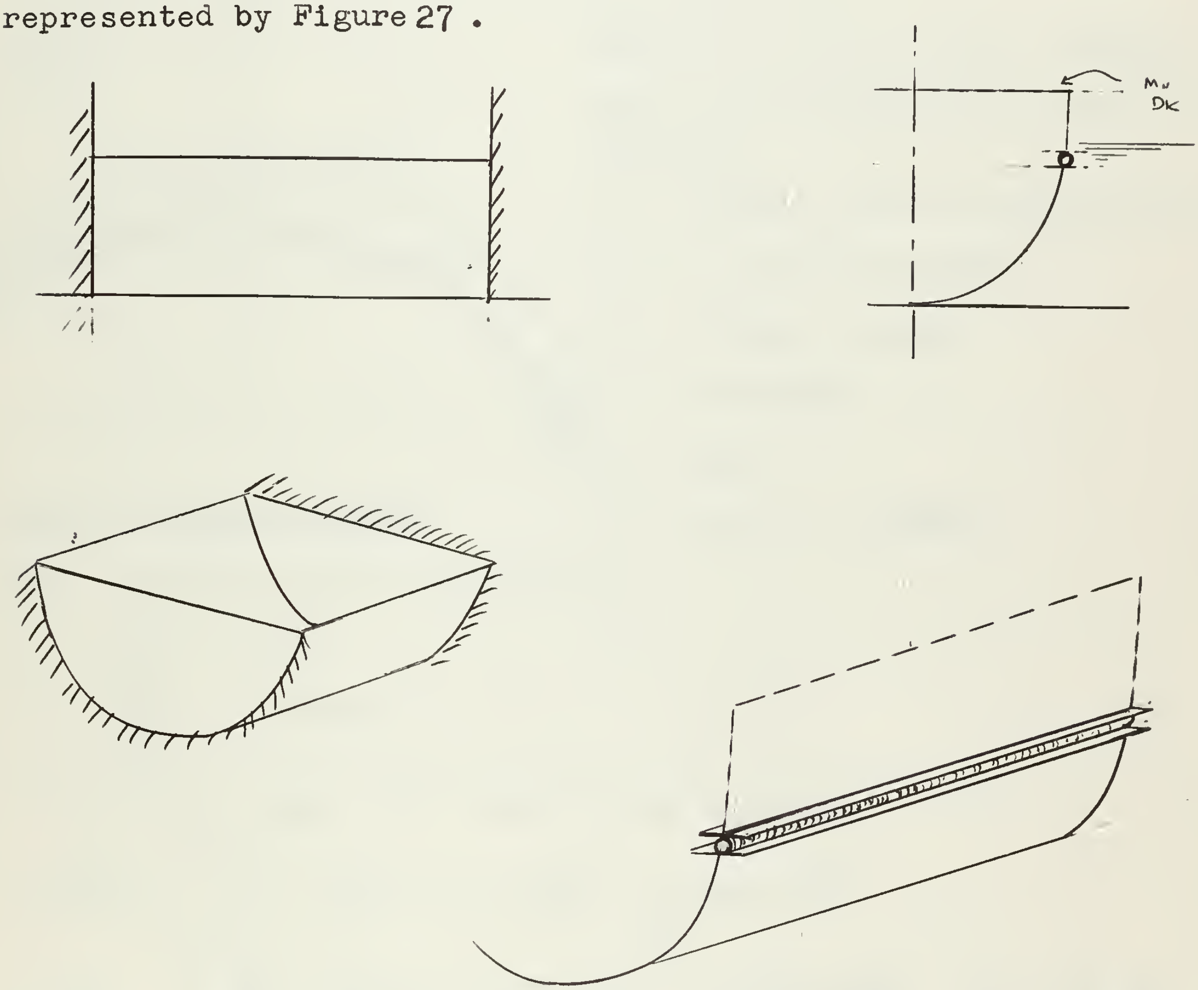



Figure 27 Degrees of Fixity at Shell Boundaries

Edge Loading;

To begin the estimate of edge loading and its distribution, consider the bottom section and the total vertical force on it due to the water (q) and the internal loading (w).

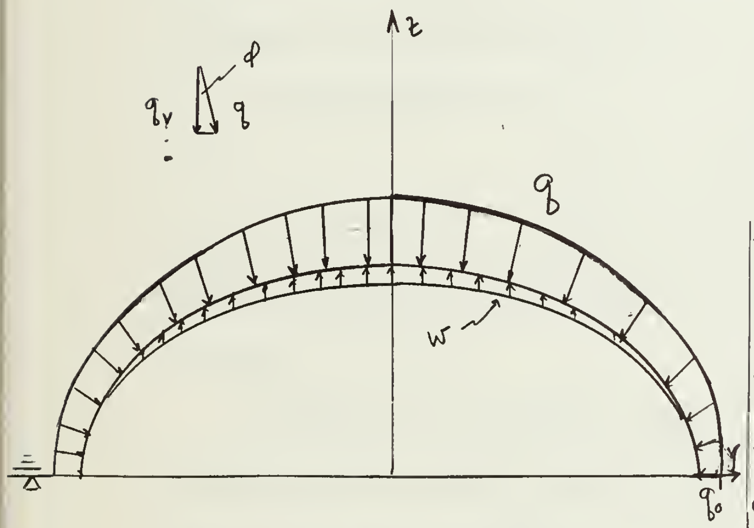



Fig. 28 Shell Loading

$$q = q_o + \gamma z$$

where $q_o = \gamma h_o$ , $h_o$ = a design head for protection of freeboard plating.

$w \approx$ const. across section (lb/FT²)

$$q_v = (q_o + \gamma z)\cos\phi$$

$$q_v = \gamma(h_o + z)\cos\phi$$

$$r = 2H \cos\phi$$

$$dS = rLd\phi$$

$$z = H/2\ (1 + \cos 2\phi)$$

total vertical force on hull $(V_t)$

$$V_t = \int (q_v - w)dS$$

$$= 2\int_0^{\pi/2} (q_v - w)rLd\phi$$

$$= 2\int_0^{\pi/2} (\gamma h_o \cos\phi - \gamma z \cos\phi - w)2H \cos\phi\ Ld\phi$$

$$= 4HL\gamma \int_0^{\pi/2} (h_o \cos^2\phi + z \cos^2\phi - \frac{w}{\gamma}\cos\phi)\ d\phi$$

$$= 4HL \int_0^{\pi/2} \left[h_o \cos^2\phi + H/2(\cos^2\phi + \cos^2\phi\cos 2\phi) - \frac{w}{\gamma}\cos\phi\right] d\phi$$

$$V_t = HL\gamma\left[\pi h_o + \frac{3\pi}{4} H - 4\frac{w}{\gamma}\right]$$

An alternate expression for $V_t$ could have been,

$$V_t = \int (q - w_n - w_t)\, dS_z$$

where $w_n$, $w_t$ are normal and tangential components of the internal loading w. $dS_z$ is an increment of surface area projection on the horizontal plane.

Substitution of

$$w_n = w \cos\theta$$
$$w_t = w \sin\theta$$
$$dS_z = r\, L \cos\phi\, d\phi$$

and integration will give same results obtained previously for $V_t$.

Having now obtained the total upward force, a determination must now be made of the distribution of the reaction forces which support $V_t$ and act on the shell boundaries. This is a statically indeterminate problem. In order to circumvent this obstacle and avoid an auxiliary analysis (whose accuracy would be greater than justified), the following approach is taken.

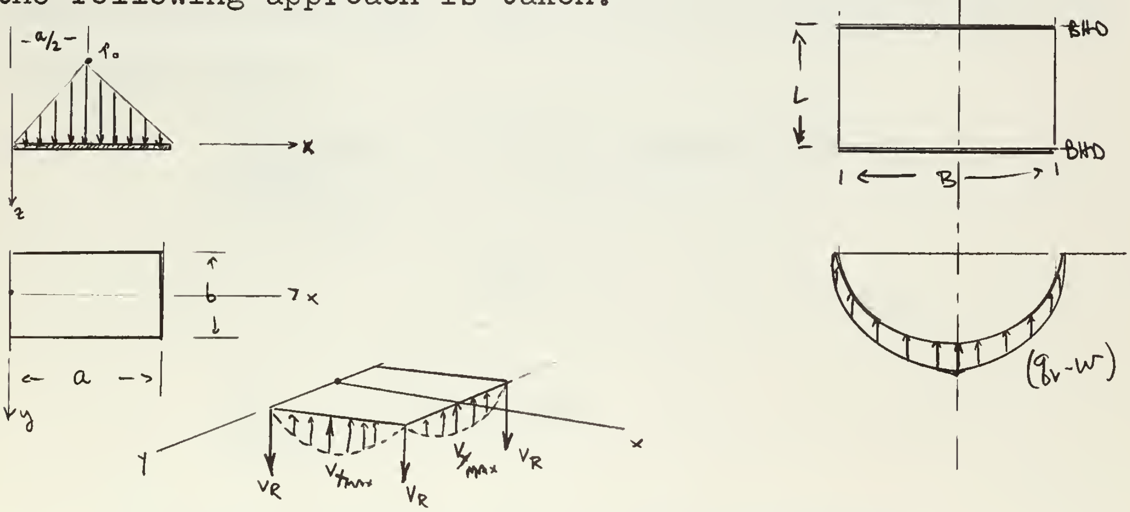



Fig. 29 Analogy of Flat Plate Prismatic Loading to Hull Loading

Timoshenko (10) describes the behavior of a simply supported rectangular flat plate under a triangular prism distribution of loading, Figure 29. The portion of this reference which is of interest is the nature of the reactive forces along the simply supported edges. Indeed it could be said that the vertical component of loading on the ships hull is a triangular prism distribution (with max value at the keel) if the hull were developed out into a flate plate, Figure 29. Because the ships loading is not on a flat plate but on a cylindrical hull, no further resemblance or analogy will be made other than a consideration of the relative distribution of the reactive forces and their relationship to the aspect ratio of the simple supports. Timoshenko tabulates $V_{y\ max}$, $V_{x\ max}$ for various aspect ratio of simple support boundaries. For the ship types being considered in this analysis, the pertinent ratio of B/L is approximately 1.3. This value gives

$$\frac{V_{y\ max}}{V_{x\ max}} = 2.73 \times \text{aspect ratio}$$

or

$$\frac{V_L}{V_B} = 2.73\ L/B$$

Therefore, $V_L, V_B = \frac{lb}{ft}$

$$2LV_L + 2\,BV_B = V_T$$

$$2LV_L + 2\,BV_B = HL\gamma\left(\pi h_0 + 3/4\pi\,H - \frac{4w}{\gamma}\right)$$

and,

$$\frac{V_L}{V_B} = 2.7\;L/B$$

Solving for the reaction forces gives,

$$\begin{cases} V_L = \dfrac{1.35\;BHL^2\gamma}{2.7\;L^2 + B^2}\left(\pi h_0 + \dfrac{3\pi}{4}\;H - 4\,\dfrac{w}{\gamma}\right) \quad lb/FT \\[2ex] V_B = \dfrac{0.5\;BHL\gamma}{2.7\;L^2+B^2}\left(\pi h_0 + \dfrac{3\pi}{4}\;H - \dfrac{4w}{\gamma}\right) \end{cases}$$

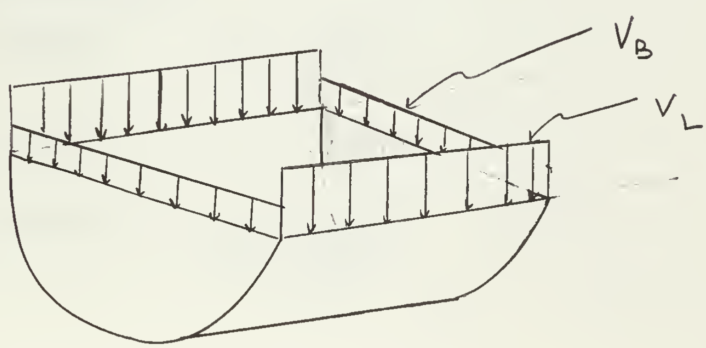



Fig. 30 Edge Loading on Shell Boundaries

## D. RIGIDITY COEFFICIENTS DATA

The orthotropic rigidity coefficients of equation (14) can be written in the following form for $\nu = 0.3$.

Extensional coefficients

$$\frac{D_x}{E} = 1.1t + \frac{A_x}{b_1}$$

$$\frac{D_\phi}{E} = 1.1t + \frac{A_\phi}{b_2}$$

$$\frac{D_\nu}{E} = 0.33t$$

$$\frac{D_{x\phi}}{E} = 0.38t$$

Flexural coefficients

$$\frac{K_x}{E} = .0915t^3 + \frac{I_x \quad A_x C_x^2}{b_2}$$

$$\frac{K_\phi}{E} = .0915t^3 + \frac{I_\phi \quad A_\phi C_\phi^2}{b_1}$$

$$\frac{K_{x\phi}}{E} = .0641t^3 + \frac{.38\ J_x}{b_2}$$

$$\frac{K_{\phi x}}{E} = .0641t^3 + \frac{.38\, J\phi}{b_1}$$

$$\frac{K_\nu}{E} = .0275t^3$$

<u>Rigidity moments</u>

$$\frac{S_x}{E} = \frac{A_x C_x}{b_2}$$

$$\frac{S_\phi}{E} = \frac{A_\phi C_\phi}{b_1}$$

TABLE 3 DD 931 RIGIDITY COEFFICIENTS DATA

| $\phi$ | $0^\circ$ | $10^\circ$ | $30^\circ$ | $45^\circ$ | $72^\circ$ | $90^\circ$ |
|---|---|---|---|---|---|---|
| (in) $t$ | .75 | .75 | .625 | .4375 | .3125 | .375 |
| ($in^2$) $A_x$ | 19.8 | 2.53 | 2.53 | 2.53 | 2.04 | 2.04 |
| (in) $b_2$ | 38.4 | 38.4 | 38.4 | 38.4 | 28.8 | 28.8 |
| (in) $C_x$ | 17.03 | 6.73 | 6.66 | 6.57 | 4.34 | 4.37 |
| ($in^4$) $I_x$ | 1104. | 26.32 | 26.32 | 26.32 | 7.61 | 7.61 |
| ($in^4$) $J_x$ | 1215 | 27.37 | 27.37 | 27.37 | 8.81 | 8.81 |
| ($in^2$) $A_\phi$ | 13.5 | 12.75 | 8.7 | 8.7 | 5.61 | 5.24 |
| (in) $C_\phi$ | 17.83 | 16.38 | 12.66 | 12.57 | 9.66 | 9.40 |
| ($in^4$) $I_\phi$ | 859.5 | 650.5 | 328.5 | 328.5 | 114.9 | 107.8 |
| ($in^4$) $J_\phi$ | 931.5 | 722.5 | 344.5 | 344.5 | 121.7 | 111.8 |

(in) $b_1 = 84 =$ const.

## TABLE 4 DD 931 CALCULATED EXTENSIONAL RIGIDITY COEFFICIENTS AT BULKHEAD

| $\phi$ | 0° | 10° | 30° | 45° | 60° | 72° | 80° |
|---|---|---|---|---|---|---|---|
| $N_x$ | +13936 | +13341 | +9150 | +3200 | -3730 | +4800 | 148000 |
| $N_\phi$ | -35805 | -34360 | -24950 | -15,380 | -7360 | - 3224 | - 1479 |
| $N_{x\phi}$ | 0 | 10753 | 27910 | 33920 | 33800 | 32690 | 38850 |
| $\sigma_1$ | 16600 | 16220 | 13450 | 10470 | 7300 | 5310 | 4480 |
| $t_e$ | 1.215 | 1.08 | 0.58 | 0.36 | 0.308 | 0.293 | 0.88 |
| $\frac{D_x}{E} \approx \frac{D_\phi}{E}$ | 1.339 | 1.19 | .638 | .396 | .339 | .322 | .969 |
| $\frac{D_\nu}{E}$ | 0.4 | 0.356 | 0.191 | 0.119 | 0.102 | .096 | 0.29 |
| $\frac{D_{x\phi}}{E}$ | 0.461 | 0.41 | 0.22 | 0.137 | 0.117 | 0.111 | 0.334 |

TABLE 5 DD 931 RIGIDITY COEFFICIENTS (Actual)

| $\phi$ | $0^\circ$ | $10^\circ$ | $30^\circ$ | $45^\circ$ | $72^\circ$ | $90^\circ$ |
|---|---|---|---|---|---|---|
| $\frac{D_x}{E}$ | 1.341 | 0.891 | 0.754 | 0.547 | 0.415 | 0.484 |
| $\frac{D_\phi}{E}$ | 0.986 | 0.977 | 0.792 | 0.585 | 0.411 | 0.475 |
| $\frac{D_\nu}{E}$ | 0.248 | 0.248 | 0.206 | 0.144 | 0.103 | 0.124 |
| $\frac{D_{x\phi}}{E}$ | 0.285 | 0.285 | 0.238 | 0.166 | 0.119 | 0.143 |
| $\frac{K_x}{E}$ | 178.8 | 3.70 | 3.64 | 3.54 | 1.60 | 1.62 |
| $\frac{K_\phi}{E}$ | 61.45 | 48.39 | 20.63 | 20.36 | 7.56 | 6.7 |
| $\frac{K_\nu}{E}$ | 0.0115 | 0.0115 | 0.0067 | 0.0023 | 0.0008 | 0.0015 |
| $\frac{K_{x\phi}}{E}$ | 12.03 | 0.298 | 0.287 | 0.276 | 0.118 | 0.120 |
| $\frac{K_{\phi x}}{E}$ | 4.237 | 3.297 | 1.576 | 1.565 | 0.552 | 0.539 |
| $\frac{S_x}{E}$ | 8.8 | 0.44 | 0.44 | 0.434 | 0.308 | 0.310 |
| $\frac{S_\phi}{E}$ | 2.87 | 2.48 | 1.32 | 1.31 | 0.64 | 0.58 |

TABLE 6 DE 1033 CALCULATED EXTENSIONAL RIGIDITY

COEFFICIENTS AT BULKHEAD

| $\phi$ | $0^\circ$ | $10^\circ$ | $30^\circ$ | $45^\circ$ | $60^\circ$ | $72^\circ$ | $80^\circ$ |
|---|---|---|---|---|---|---|---|
| $\left(\frac{lb}{ft}\right)$ $N_x$ | +7725 | +7414 | +5201 | +2180 | -750 | +8000 | +110,000 |
| $\left(\frac{lb}{Ft}\right)$ $N_\phi$ | -24,580 | -23650 | -17330 | -10880 | -5375 | -2463 | -1162 |
| $\left(\frac{lb}{Ft}\right)$ $N_{x\phi}$ | 0 | 6463 | 16910 | 20,770 | 21,250 | 21,600 | 27,200 |
| (psi) $\sigma_1$ | 15000 | 14680 | 12150 | 9450 | 6600 | 4800 | 4050 |
| (in) $t_\theta$ | 0.538 | 0.495 | 0.321 | 0.240 | 0.198 | 0.202 | 0.635 |
| $\frac{D_x}{E} \approx \frac{D_\phi}{E}$ | 0.592 | 0.545 | 0.354 | 0.264 | 0.218 | 0.222 | 0.7 |
| $\frac{D_\nu}{E}$ | 0.1775 | 0.103 | 0.106 | 0.0792 | 0.0654 | 0.666 | .21 |
| $\frac{D_{x\phi}}{E}$ | 0.204 | 0.188 | 0.122 | 0.091 | 0.0753 | .0768 | 0.242 |

TABLE 7 DE 1033 RIGIDITY COEFFICIENTS (Actual)

| $\phi$ | $0^\circ$ | $10^\circ$ | $30^\circ$ | $45^\circ$ | $60^\circ$ | $72^\circ$ | $80^\circ$ |
|---|---|---|---|---|---|---|---|
| t | 0.5 | 0.5 | 0.4375 | 0.4375 | 0.375 | 0.375 | 0.3125 |
| $A_x$ | 24.71 | 2.81 | 2.49 | 2.20 | 2.20 | 2.20 | 2.20 |
| $b_2$ | 26.9 | 26.3 | 27.5 | 25.6 | 26.3 | 28.1 | 26.2 |
| $A_\phi$ | 8.5 | 7.3 | 4.63 | 4.57 | 4.27 | 4.03 | 3.91 |
| $\frac{D_x}{E}$ | 1.47 | 0.657 | 0.571 | 0.566 | 0.497 | 0.489 | 0.428 |
| $\frac{D_\phi}{E}$ | 0.644 | 0.631 | 0.531 | 0.53 | 0.46 | 0.458 | 0.387 |
| $\frac{D_\nu}{E}$ | 0.165 | 0.165 | 0.144 | 0.144 | 0.124 | 0.124 | 0.103 |
| $\frac{D_{x\phi}}{E}$ | 0.19 | 0.19 | 0.166 | 0.166 | 0.142 | 0.142 | 0.119 |
| $b_1 = 90'' =$ const. | | | | | | | |

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